

On form factors of boundary changing operators

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Abstract

We develop a form factor bootstrap program to determine the matrix elements of local, boundary condition changing operators. We propose axioms for these form factors and determine their solutions in the free boson and Lee-Yang models. The sudden change in the boundary condition, caused by an operator insertion, can be interpreted as a local quench and the form factors provide the overlap of any state before the quench with any outgoing state after the quench.

1 Introduction

Integrable 1+1 dimensional systems are very special quantum field theories as they can be solved exactly [1, 2]. The models and the obtained solutions are interesting in many respects. First, they appear on various areas of theoretical physics ranging from statistical physics to string theory. Second, the exact solutions can be compared to and test alternative approximate solutions.

The procedure of solving integrable theories consists of two steps. In the first step the scattering (S) and reflection (R) matrices, connecting asymptotic initial and final states, are determined. These contain the on-shell information of a given bulk or boundary quantum field theory. In the second step restrictive functional equations are formulated for the form factors involving the already determined S and R matrices. The solutions of these equations provide off-shell information which then can be used to calculate the correlation functions via the spectral representation.

Recently there have been increasing interest in quench type problems. They appear when, at a given time, a parameter of the physical system is changed. They are relevant in statistical physics and solid state problems. On the string theory side they appear when the strings split, fuse or change their boundary conditions [3, 4]. So far the integrable approaches assumed a squeezed coherent (boundary) state form of the system after the quench, see [5, 6, 7] and references therein. Contrary, we would like to analyze a different quench, which is related to form factors. As an example let us suppose that we introduce a quench in a system at a moment by inserting a local operator \mathcal{O} , which we can even integrate in space $\int \mathcal{O}(x, 0) dx$.

In the quench framework we are interested in how a given state (say the vacuum) will evolve after the quench. This is probed by the matrix elements

$$\langle \theta_1, \dots, \theta_n | \int \mathcal{O}(x, 0) dx | 0 \rangle = F^{\mathcal{O}}(\bar{\theta}_n, \dots, \bar{\theta}_1) \delta_P \quad , \quad \bar{\theta} = \theta + i\pi \quad , \quad (1.1)$$

which is basically the form factor of the operator \mathcal{O} , and δ_P projects onto zero momentum states. Clearly, form factors do not exponentiate, except from free theories. This quench is, however, localized in time, and cannot be regarded as a change of a parameter of the model.

In the following we will be interested in another integrable quench, which changes the parameters of the theory but still corresponds to form factors. We analyze an integrable boundary system in which at a moment we change the integrable boundary condition from α to β by inserting a boundary condition changing operator. These kinds of boundary quenches have been used to calculate the Loschmidt echo in the Resonant Level Model [8]. As the vacuum evolves to the form factors of the boundary condition changing operator we formulate axioms to determine these quantities.

In [9, 10] the authors proposed form factor axioms both for boundary operators and for boundary changing operators. First they adopted the boundary form factor axioms from lattice models [11] and adjusted them for the relativistic kinematics. Then, on the example of the free massive fermion model they generalized them for operators which change the boundary condition and they further analyzed the solutions of these equations. Finally, they extended the axioms for non-trivial bulk scatterings and investigated the sinh-Gordon model, where they calculated the form factors of boundary changing operators up to 4 particles. They also extended the analysis for massless scatterings and applied the results for the double well problem of dissipative quantum mechanics.

In [12] the authors analyzed the form factors of local boundary operators from a different perspective. They derived a closed set of boundary form factor axioms from the boundary reduction formula [13]. These axioms, besides of the previous ones of [9], additionally contained the boundary kinematical singularity axiom, making the whole system complete in the sense, that the space of the solutions is in one to one correspondence with the space of all local boundary operators of the UV boundary conformal field theory [14]. This boundary form factor program was carried out in many integrable models and was generalized to nondiagonal scattering theories [15, 16, 17, 18].

The aim of the present paper is to extend this form factor program for boundary changing operators, i.e. our axioms, additionally to the axioms of [9], contain the boundary changing analogue of the boundary kinematical singularity axiom. We also show that our axioms are complete in the above sense, as we find as many solutions as many boundary changing local operators exists in the UV limiting boundary conformal field theory.

The paper is organized as follows: In Section 2 we introduce the theory of form factors in integrable field theories and present our proposals for the boundary changing form factor axioms. Various consistency checks are presented and we show the general method to solve them. Their applicability to the calculation of two point functions is also explained. In Section 3 we solve the axioms in case of the free boson and Lee-Yang theories. In the free boson theory direct field theoretical approach is also presented. In case of the Lee-Yang model two-point functions of boundary fields are calculated by summing up few particle form factor contributions and compared, at short distance, to the conformal field theory prediction. Their agreement is a solid confirmation of our form factor solutions. We end the main part of the paper by the conclusion in Section 4. Some technical details are relegated to the

two appendices. In Appendix A a formal derivation of the axioms from the Zamolodchikov-Faddeev algebra is shown. In Appendix B we study the free boson theory in which we change the boundary condition from Neumann to Dirichlet. Besides the bootstrap approach, direct infinite and finite volume field theoretical calculations are presented, and the relation to the open-closed string vertex [3, 4] is demonstrated.

2 Form factor axioms for boundary changing operators

In this section we formulate the axioms, which have to be satisfied by the matrix elements of local boundary condition changing operators. We start by describing an integrable boundary system with a given boundary condition, and focus on changing of the boundary condition afterward. The calculation of two point function is also considered.

2.1 Integrable boundary systems

The Hilbert space of an integrable boundary system consists of multi-particle states labeled by the particles' rapidities and their particle types. For simplicity we analyze theories containing only one particle type with a given mass m . Particles are then characterized only by their rapidities, such that their energy and momentum are

$$E = m \cosh \theta \quad , \quad p = m \sinh \theta. \quad (2.1)$$

Asymptotic *in* states are prepared in the remote past, when particles get far away from each other and from the boundary, which we put on the right of the half-space at $x = 0$. This well separated particle state is equivalent to a free multi-particle state, which we denote by

$$|\theta_1, \theta_2, \dots, \theta_n\rangle_{in}^\alpha \quad , \quad \theta_1 > \theta_2 > \dots > \theta_n > 0 \quad (2.2)$$

where α labels the boundary condition.

For $t \rightarrow +\infty$ all scatterings and reflections are terminated, the particles are again far away from each other and from the boundary forming the *out* state,

$$|\theta'_1, \theta'_2, \dots, \theta'_m\rangle_{out}^\alpha \quad , \quad \theta'_1 < \theta'_2 < \dots < \theta'_m < 0 \quad (2.3)$$

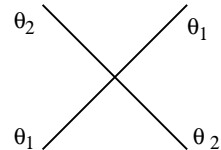
which is again equivalent to a free state. The two sets of states form a complete basis separately and are connected by the multiparticle reflection matrix. In an integrable theory, this reflection matrix factorizes into the product of pairwise bulk scatterings and individual reflections

$$|\theta_1, \theta_2, \dots, \theta_n\rangle_{in}^\alpha = \prod_{i < j} S(\theta_i - \theta_j) S(\theta_i + \theta_j) \prod_i R^\alpha(\theta_i) |-\theta_1, -\theta_2, \dots, -\theta_n\rangle_{out}^\alpha \quad (2.4)$$

where $S(\theta_i - \theta_j)$ connects the two particle asymptotic *in* and *out* states in the bulk theory

$$|\theta_1, \theta_2\rangle_{in}^{bulk} = S(\theta_1 - \theta_2) |\theta_2, \theta_1\rangle_{out}^{bulk}$$

depicted as



It is defined originally for $\theta_1 > \theta_2$ but can be analytically continued for complex rapidity parameters such that the extended function (denoted the same way) is meromorphic and satisfies unitarity and crossing symmetry

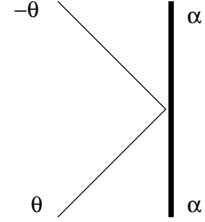
$$S(\theta)S(-\theta) = 1 \quad , \quad S(i\pi - \theta) = S(\theta) \quad (2.5)$$

It might have poles on the imaginary axis at locations $\theta = iu_j$ with residue $-i \text{res}_{\theta=iu_j} S(\theta) = \Gamma_j^2$, some of which correspond to bound states.

The amplitude $R^\alpha(\theta)$ connects the one particle asymptotic states in the boundary theory

$$|\theta\rangle_{in}^\alpha = R^\alpha(\theta) |-\theta\rangle_{out}^\alpha$$

depicted as



It can also be extended from the fundamental domain $\theta > 0$ to a meromorphic function on the whole complex θ plane satisfying unitarity and boundary crossing unitarity

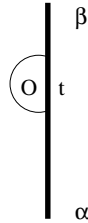
$$R^\alpha(\theta)R^\alpha(-\theta) = 1 \quad , \quad R^\alpha(i\pi - \theta)S(2\theta) = R^\alpha(\theta) \quad (2.6)$$

$R^\alpha(\theta)$ may have poles at imaginary locations $\theta = iv_j$ ($0 < v_j < \pi/2$), with residues $i\tilde{g}^2/2$, some corresponding to excited boundary states. If the interpolating field has a nontrivial vacuum expectation value then generally there is also a pole at $\theta = i\pi/2$ with residue

$$-i \text{Res}_{\theta=\frac{i\pi}{2}} R^\alpha(\theta) = \frac{g_\alpha^2}{2}. \quad (2.7)$$

2.2 Boundary changing operators

A boundary condition changing operator, $\mathcal{O}_{\beta\alpha}(t)$ is a local operator, inserted at t , which changes the boundary condition from α , valid for time smaller than t , to β , valid for times large than t . Graphically it is represented as



The form factor of this boundary condition changing operator is defined by its matrix element between asymptotic states related to the boundary conditions α and β . We expect that the Hamiltonians valid before and after the insertion can be used to transport the operator in time, such that

$$\begin{aligned} {}^\beta_{out} \langle \theta'_m, \dots, \theta'_2, \theta'_1 | \mathcal{O}_{\beta\alpha}(t) | \theta_1, \theta_2, \dots, \theta_n \rangle_{in}^\alpha &= \\ F_{mn}^{\mathcal{O}_{\beta\alpha}}(\theta'_m, \dots, \theta'_1; \theta_1, \dots, \theta_n) e^{-it(m \sum \cosh \theta_i + \Delta E_{\text{bdry}}^{\beta\alpha} - m \sum \cosh \theta'_j)} \end{aligned} \quad (2.8)$$

where the difference in the boundary energies is $\Delta E_{\text{bdry}}^{\beta\alpha} = E_\alpha - E_\beta$. From now on we focus on the t -independent form factor $F_{mn}^{\mathcal{O}\beta\alpha}$. It is defined originally for $\theta_1 > \theta_2 > \dots > \theta_n > 0$ and $\theta'_1 < \theta'_2 < \dots < \theta'_m < 0$, but can be analytically continued for any orderings and signs of the rapidities, and also for complex values.

In [12] the form factors of a local boundary operator were related to the correlation functions of the boundary theory via the boundary reduction formula. The idea of the reduction formula is that for large negative time the finite energy configurations contain well localized separated particle states being far from each other and from the boundary, thus forming an excitation of the free theory. The interaction in this limit can be switched off adiabatically and the interacting quantum field agrees with the free field up to the wave-function renormalization constant. The particle creation operator, expressed in terms of the free field, can be traded for the interpolating field and the locality of the operator insertion guaranties a domain of convergence for the continuation of the form factor in the complex rapidity plane. Applying the same procedure for an outgoing state and comparing the two expressions a crossing relation can be obtained between the two form factors. By replacing the local boundary operator with a local boundary *changing* operator the continuity of the interpolating field is not changed and similar argumentations can be applied, which leads to the crossing formula

$$F_{mn}^{\mathcal{O}\beta\alpha}(\theta'_m, \dots, \theta'_2, \theta'_1; \theta_1, \theta_2, \dots, \theta_n) = F_{m-1n+1}^{\mathcal{O}\beta\alpha}(\theta'_m, \dots, \theta'_2; \theta'_1 + i\pi, \theta_1, \theta_2, \dots, \theta_n) + \text{disc.} \quad (2.9)$$

where disc. represents disconnected terms appearing whenever θ'_1 equals any of the θ_i . As a result of this crossing transformation we can express all form factors in terms of the elementary form factors

$${}^\beta_{out} \langle 0 | \mathcal{O}(0) | \theta_1, \theta_2, \dots, \theta_n \rangle^\alpha_{in} = F_n^{\mathcal{O}\beta\alpha}(\theta_1, \theta_2, \dots, \theta_n) \quad (2.10)$$

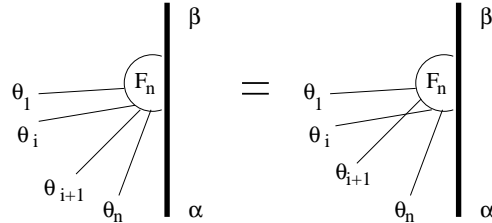
Let us note that boundary form factors $F_n^{\mathcal{O}\beta\alpha}(\theta_1, \dots, \theta_n)$ do depend in general on all the rapidities θ_i , not just on their differences, as the boundary breaks the Lorentz invariance.

2.3 Axioms for the elementary form factors

The form factor properties can be formally derived from the Zamolodchikov-Faddeev algebra, see Appendix A and also [9]. We take these properties as axioms, such that functions satisfying them determine local boundary changing operators completely.

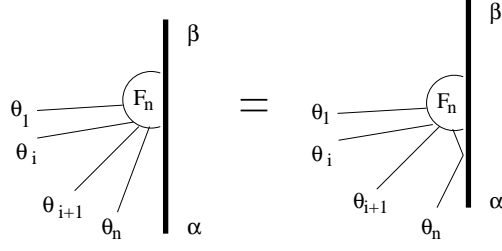
I. Permutation:

$$F_n^{\mathcal{O}\beta\alpha}(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n) = S(\theta_i - \theta_{i+1}) F_n^{\mathcal{O}\beta\alpha}(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n) \quad (2.11)$$



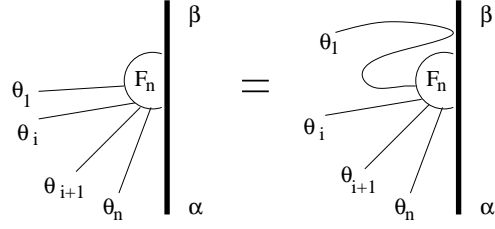
II. Reflection:

$$F_n^{\mathcal{O}\beta\alpha}(\theta_1, \dots, \theta_{n-1}, \theta_n) = R^\alpha(\theta_n) F_n^{\mathcal{O}\beta\alpha}(\theta_1, \dots, \theta_{n-1}, -\theta_n) \quad (2.12)$$



III. Crossing reflection:

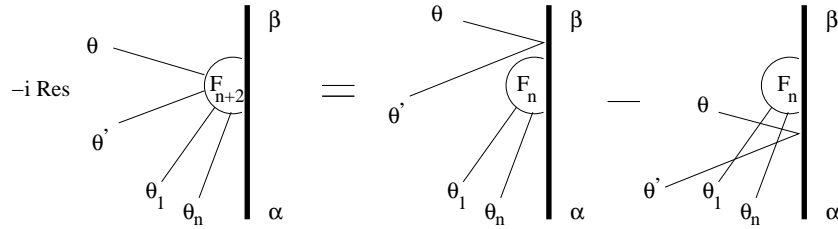
$$F_n^{\mathcal{O}\beta\alpha}(\theta_1, \theta_2, \dots, \theta_n) = R^\beta(i\pi - \theta_1) F_n^{\mathcal{O}\beta\alpha}(2i\pi - \theta_1, \theta_2, \dots, \theta_n) \quad (2.13)$$



The singularity structure of the form factors is determined on physical grounds and can be axiomatized as follows:

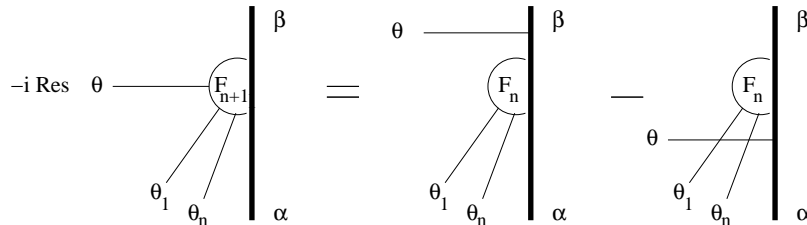
IV. Kinematical singularity:

$$-i \operatorname{Res}_{\theta=\theta'} F_{n+2}^{\mathcal{O}\beta\alpha}(-\theta + i\pi, \theta', \theta_1, \dots, \theta_n) = \left(R^\beta(\theta) - \prod_{i=1}^n S(\theta - \theta_i) R^\alpha(\theta) S(\theta + \theta_i) \right) F_n^{\mathcal{O}\beta\alpha}(\theta_1, \dots, \theta_n) \quad (2.14)$$



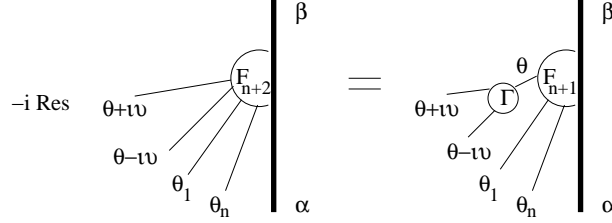
V. Boundary kinematical singularity:

$$-i \operatorname{Res}_{\theta=0} F_{n+1}^{\mathcal{O}\beta\alpha}(\theta + \frac{i\pi}{2}, \theta_1, \dots, \theta_n) = \left(\frac{g_\beta}{2} - \frac{g_\alpha}{2} \prod_{i=1}^n S(\frac{i\pi}{2} - \theta_i) \right) F_n^{\mathcal{O}\beta\alpha}(\theta_1, \dots, \theta_n) \quad (2.15)$$



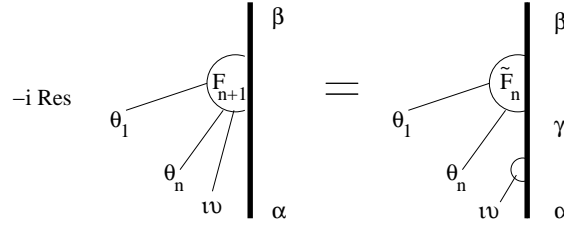
VI. Bulk dynamical singularity:

$$-i \operatorname{Res}_{\theta=\theta'} F_{n+2}^{\mathcal{O}_{\beta\alpha}}(\theta + iu, \theta' - iu, \theta_1, \dots, \theta_n) = \Gamma F_{n+1}^{\mathcal{O}_{\beta\alpha}}(\theta, \theta_1, \dots, \theta_n) \quad (2.16)$$



VII. Boundary dynamical singularity:

$$-i \operatorname{Res}_{\theta=iv} F_{n+1}^{\mathcal{O}_{\beta\alpha}}(\theta_1, \dots, \theta_n, \theta) = \tilde{g}_\alpha \tilde{F}_n^{\mathcal{O}_{\beta\alpha}}(\theta_1, \dots, \theta_n). \quad (2.17)$$



We would like to remark here that the axioms, except the boundary kinematical singularity axiom, has already been proposed in [9], derived from the Zamolodchikov-Faddeev algebra, in a similar fashion as presented in Appendix A. The boundary kinematical singularity axiom is crucial as it can differentiate between physically different boundary condition having the same reflection factor but different sign of g .

2.4 Consistency checks

First we note that these axioms reduce to the form factor axioms of local boundary operators in the $\alpha = \beta$ case. Furthermore, we can also perform the same consistency checks, which were done for the boundary form factors in [12]. Let us note that the axioms are self-consistent in the sense that for specific rapidities the $n + 2$ particle form factor can be connected to the n particle form factor either by the kinematical singularity equations or by using twice the boundary kinematical equations, and the two procedures give the same result. Indeed taking double residue in the first case, first at $\theta = \theta'$ and then at $\theta = i\frac{\pi}{2}$ gives

$$\begin{aligned} i \operatorname{Res}_{\theta=i\frac{\pi}{2}} i \operatorname{Res}_{\theta'=\theta} F_{n+2}^{\mathcal{O}_{\beta\alpha}}(-\theta + i\pi, \theta', \theta_1, \dots, \theta_n) &= \\ &= \left(-i \operatorname{Res}_{\theta=i\frac{\pi}{2}} \right) \left(R^\beta(\theta) - R^\alpha(\theta) \prod_{i=1}^n S\left(\frac{i\pi}{2} - \theta_i\right) S\left(\frac{i\pi}{2} + \theta_i\right) \right) F_n^{\mathcal{O}_{\beta\alpha}}(\theta_1, \dots, \theta_n). \end{aligned} \quad (2.18)$$

Taking now the residue at $\theta = \frac{i\pi}{2}$ first then at $\theta' = \frac{i\pi}{2}$ and using that $S(0) = -1$ gives

$$\begin{aligned} i \operatorname{Res}_{\theta=\frac{i\pi}{2}} i \operatorname{Res}_{\theta'=\frac{i\pi}{2}} F_{n+2}^{\mathcal{O}_{\beta\alpha}}(-\theta + i\pi, \theta', \theta_1, \dots, \theta_n) &= \\ &= \left(\frac{g_\beta}{2} + \frac{g_\alpha}{2} \prod_{i=1}^n S\left(\frac{i\pi}{2} - \theta_i\right) \right) \left(\frac{g_\beta}{2} - \frac{g_\alpha}{2} \prod_{i=1}^n S\left(\frac{i\pi}{2} - \theta_i\right) \right) F_n^{\mathcal{O}_{\beta\alpha}}(\theta_1, \dots, \theta_n). \end{aligned} \quad (2.19)$$

Combining the crossing symmetry of the S-matrix with the definition of g (2.7) the two expressions are easily seen to be equivalent.

There is another consistency check of the axioms, if one of the boundary conditions can be obtained from the other by binding a particle to it. This does not necessarily mean a boundary bound-state form factor, as many boundary conditions can be obtained by placing an integrable defect in front of a boundary [19]. If R^β denotes the reflection factor of an integrable boundary condition and $T_\pm(\theta)$ the left/right transmission factor of an integrable defect then the reflection factor of the dressed boundary is

$$R^\alpha(\theta) = T_-(\theta) R^\beta(\theta) T_+(\theta) \quad (2.20)$$

A particle with imaginary rapidity, θ_0 , can always be considered as an integrable defect $T_\mp(\theta) = S(\theta \mp \theta_0)$ and in this case the dressed boundary reflection factor is

$$R^\alpha(\theta) = S(\theta - \theta_0) S(\theta + \theta_0) R^\beta(\theta) \quad (2.21)$$

which formally looks like a boundary excited reflection factor. One example for this situation is the scaling Lee-Yang model with integrable boundary conditions. There are two types of boundary conditions: the $\beta = \mathbb{I}$ identity boundary condition, which does not allow any bound-state and the $\alpha = \Phi$ boundary condition, which carries a label b , and can be realized in the above sense with $\theta_0 = \frac{i\pi(3-b)}{6}$. The implementation of binding a particle with rapidity θ_0 to the β boundary is to consider the form factor equations for $F_{n+1}^{\mathcal{O}_{\beta\beta}}(\theta_1, \dots, \theta_n, \theta_0)$ in the rapidities $\theta_1, \dots, \theta_n$ only. We claim that the equations are the same as we presented above for $F_n^{\mathcal{O}_{\beta\alpha}}(\theta_1, \dots, \theta_n)$. The permutation and crossing reflection axioms are trivially the same. For the reflection axiom we move θ_n through θ_0 , use the reflection axiom of the $\beta = \alpha$ case and move back the reflected $-\theta_n$ through θ_0 . As a result we obtain the dressed reflection factor (2.21). The singularity axioms can easily be seen to be the same, too. Let us note that although all the equations for $F_n^{\mathcal{O}_{\beta\alpha}}(\theta_1, \dots, \theta_n)$ appear as equations for $F_{n+1}^{\mathcal{O}_{\beta\beta}}(\theta_1, \dots, \theta_n, \theta_0)$, the latter one satisfies additional axioms, such as the permutation or reflection axiom involving θ_0 , thus we do not expect the two form factors to be equal.

2.5 General solution of the form factor axioms

We start this section by determining the one particle form factor and use later this solution to construct the general multiparticle form factor. In order to simplify notations we suppress the operator $\mathcal{O}_{\beta\alpha}$ in the index of the form factor and write only $\beta\alpha$ explicitly.

The equations for the one particle form factor read¹:

$$F_1^{\beta\alpha}(\theta) = R^\alpha(\theta) F_1^{\beta\alpha}(-\theta) \quad ; \quad F_1^{\beta\alpha}(i\pi + \theta) = R^\beta(-\theta) F_1^{\beta\alpha}(i\pi - \theta), \quad (2.22)$$

¹These equations had been also found in [9] in the context of the sinh-Gordon theory

where the reflection amplitudes $R^\alpha(\theta)$, $R^\beta(\theta)$ are assumed to be meromorphic. From general considerations we assume that $F_1^{\beta\alpha}(\theta)$ is analytic on $0 \leq \Im(\theta) \leq \pi$. The construction of solving (2.22) is reduced to a problem already solved in the bulk form factor bootstrap. To this end we write

$$F_1^{\beta\alpha}(\theta) = h^\alpha(\theta)h^\beta(i\pi - \theta) \quad (2.23)$$

and suppose that

$$h^\gamma(\theta) = R^\gamma(\theta)h^\gamma(-\theta) \quad , \quad h^\gamma(i\pi + \theta) = h^\gamma(i\pi - \theta) \quad , \quad \gamma = \alpha, \beta \quad (2.24)$$

which are nothing else but the bulk two particle form factor equations [20], where the reflection amplitude, $R^\gamma(\theta)$, plays the role of the S-matrix. To obtain a solution of (2.24) we use the theorem of Karowski and Weisz [20]. Assume that the function $h(\theta)$ is meromorphic in the physical strip $0 \leq \Im(\theta) < \pi$ with possible poles at $i\alpha_1, \dots, i\alpha_l$ and zeros at $i\beta_1, \dots, i\beta_k$ and grows as at most a polynomial in $\exp(|\theta|)$ for $|\Re \theta| \rightarrow \infty$, furthermore it satisfies

$$h(\theta) = R(\theta)h(-\theta) \quad , \quad h(i\pi - \theta) = h(i\pi + \theta) \quad , \quad R(\theta) = \exp \left\{ \int_0^\infty dt f(t) \sinh \left(\frac{t\theta}{i\pi} \right) \right\} \quad (2.25)$$

then it is uniquely defined up to normalization as

$$h(\theta) = \frac{\prod_{j=1}^k \sinh \left(\frac{1}{2}(\theta - i\beta_j) \right) \sinh \left(\frac{1}{2}(\theta + i\beta_j) \right)}{\prod_{j=1}^l \sinh \left(\frac{1}{2}(\theta - i\alpha_j) \right) \sinh \left(\frac{1}{2}(\theta + i\alpha_j) \right)} \exp \left\{ \int_0^\infty dt f(t) \frac{\sin^2 \left(\frac{i\pi - \theta}{2\pi} t \right)}{\sinh t} \right\}. \quad (2.26)$$

In the typical applications the reflection amplitude can be expressed as products of the blocks, (x_i) ,

$$R^\gamma(\theta) = \prod_{i=1}^k (x_i^\gamma) \quad , \quad -(x) = -\frac{\sinh \left(\frac{\theta}{2} + i\frac{\pi x}{2} \right)}{\sinh \left(\frac{\theta}{2} - i\frac{\pi x}{2} \right)} = \exp \left\{ 2 \int_0^\infty \frac{dt \sinh t (1-x)}{t \sinh t} \sinh \left(\frac{t\theta}{i\pi} \right) \right\} \quad (2.27)$$

where $0 < x < 1$. The validity of this integral representation can be extended by periodicity $(x \pm 2) = (x)$ and by the relation $(-x) = (x)^{-1}$. Thus the minimal solution, corresponding to $(-1)^k R^\gamma(\theta)$ is given as

$$h^\gamma(\theta) = \exp \left\{ 2 \int_0^\infty \frac{dt \sum_{i=1}^k \sinh (t(1 - x_i^\gamma))}{t \sinh^2 t} \sin^2 \left(\frac{i\pi - \theta}{2\pi} t \right) \right\} \quad (2.28)$$

if k is even. In case of odd k , due to the extra minus sign in $R^\gamma(\theta)$, the minimal solution $h^\gamma(\theta)$ necessarily contains a zero at the origin which can be implemented by putting an extra $\sinh \frac{\theta}{2}$ into it.

We would like to remark that the one-particle minimal boundary changing form factors (2.22) have been found, by slightly different methods, in the off-critical Ising model, the sinh-Gordon model and for double well problem of dissipative quantum mechanics [10, 9].

Note that if $F_1^{\beta\alpha}(\theta)$ is a solution of (2.22) then $F_1^{\beta\alpha}(\theta)Q(\theta)$ is also a solution provided $Q(\theta) = Q(-\theta)$ and $Q(i\pi + \theta) = Q(i\pi - \theta)$, i.e. if Q is even and $2\pi i$ periodic. Therefore one can assume that Q is the function of $y = e^\theta + e^{-\theta}$. Thus the general solution of eq. (2.22) can be written as

$$F_1^{\beta\alpha}(\theta) = r^{\beta\alpha}(\theta)Q_1(y), \quad y = e^\theta + e^{-\theta}, \quad (2.29)$$

The general form of the multi-particle form factors which, additionally to the reflection equations, satisfies also the permutation and the singularity equations, can be written in the following form²:

$$F_n^{\beta\alpha}(\theta_1, \theta_2, \dots, \theta_n) = H_n \prod_{i=1}^n \frac{r^{\beta\alpha}(\theta_i)}{y_i} \prod_{i<j} \frac{f(\theta_i - \theta_j) f(\theta_i + \theta_j)}{(y_i + y_j)} Q_n(y_1, y_2, \dots, y_n). \quad (2.30)$$

Here $f(\theta)$ is the minimal bulk two particle form factor, defined as the minimal solution, i.e. the one with the least poles and zeros compatible with the dynamics of the theory, of the equations

$$f(\theta) = S(\theta) f(-\theta) \quad , \quad f(i\pi - \theta) = f(i\pi + \theta). \quad (2.31)$$

As a consequence of the form factor equations, Q_n is a $2\pi i$ periodic, symmetric and even function of the rapidities, θ_i , i.e. it is symmetric in the variable $y_i = 2 \cosh \theta_i$. The denominator $\prod_i y_i$ is responsible for the boundary, while the product $\prod_{i<j} (y_i + y_j)$ for the bulk kinematical singularity. The boundary and bulk kinematical singularity axioms result in recursions relating Q_n to Q_{n-1} and Q_{n-2} , respectively. The bulk dynamical pole equation relates also Q_n to Q_{n-1} if it is present. The corresponding pole is usually included in $f(\theta)$.

An important restriction on the form factor functions follows from requiring a power law bounded ultraviolet behaviour for the two point correlator of two boundary changing operators $\langle 0 | \mathcal{O}^{\gamma\beta}(\tau) \mathcal{O}^{\beta\alpha}(0) | 0 \rangle$: the growth of the function $F_n^{\beta\alpha}(\theta_1, \dots, \theta_n)$ must be bounded by some exponential of the rapidity as $\theta \rightarrow \infty$ (i.e. the form factors only grow polynomially with particle energy). If $r(\theta)$ and $f(\theta)$ are specified in a way to include all poles induced by the dynamics of the model, then it follows that the functions Q_n must be *polynomials* of the variables y_i .

2.6 Two-point function

Once an appropriate solution of the form factor axioms is found, it can be used to describe correlators of boundary changing operators. The two-point function of boundary changing operators can be computed by inserting a complete set of states

$$\langle 0 | \mathcal{O}^{\gamma\beta}(t) \mathcal{O}^{\beta\alpha}(0) | 0 \rangle = \sum_{n=0}^{\infty} \frac{1}{(2\pi)^n} \int_{\theta_1 > \theta_2 > \dots > \theta_n > 0} d\theta_1 d\theta_2 \dots d\theta_n e^{-it\Delta E_{\text{bdry}}^{\gamma\beta} - imt \sum_i \cosh \theta_i} F_n^{\gamma\beta} F_n^{\beta\alpha+} \quad (2.32)$$

where time translation covariance was used, and the form factors were abbreviated by

$$F_n^{\beta\alpha} = \langle 0 | \mathcal{O}^{\beta\alpha}(0) | \theta_1, \theta_2, \dots, \theta_n \rangle_{in} = F_n^{\beta\alpha}(\theta_1, \theta_2, \dots, \theta_n) \quad (2.33)$$

and by

$$F_n^{\beta\alpha+} = {}_{in} \langle \theta_1, \theta_2, \dots, \theta_n | \mathcal{O}^{\beta\alpha}(0) | 0 \rangle = F_n^{\beta\alpha}(i\pi + \theta_n, i\pi + \theta_{n-1}, \dots, i\pi + \theta_1). \quad (2.34)$$

The latter one, for unitary theories, is the complex conjugate of the first one: $F_n^+ = F_n^*$. In the Euclidean ($r = it$) version of the theories the form factor expansion of the correlator for large separations converges rapidly since multi-particle terms are exponentially suppressed.

²This parametrization was found also in [9] for the off-critical Ising and sinh-Gordon model.

3 Model studies

In this section we explicitly carry out the form factor bootstrap program in the free boson and Lee-Yang models.

3.1 Free boson with linear boundary conditions

As a first step we carry out the form factor bootstrap program and calculate explicitly the form factors of the operators, which change the linear boundary condition with parameter λ^α to that of with λ^β . When the boundary is changed from Neumann to Dirichlet we recover the same result from the direct solution of the model.

3.1.1 Solution of the form factor equation

The reflection factor of the free boson with linear boundary condition has the form

$$R^\gamma(\theta) = \frac{\sinh \theta - i\lambda^\gamma}{\sinh \theta + i\lambda^\gamma} \quad (3.1)$$

Following the general strategy, we search for the one particle form factor $F_1^{\beta\alpha}(\theta)$ in the form

$$F_1^{\beta\alpha}(\theta) = r^{\beta\alpha}(\theta)Q_1(y) \quad , \quad r^{\beta\alpha}(\theta) = h^\alpha(\theta)h^\beta(i\pi - \theta) \quad , \quad y = e^\theta + e^{-\theta} \quad (3.2)$$

where the functions $h^\gamma(\theta)$ satisfy

$$h^\gamma(\theta) = R^\gamma(\theta)h^\gamma(-\theta) \quad , \quad h^\gamma(i\pi + \theta) = h^\gamma(i\pi - \theta) \quad , \quad \gamma = \alpha, \beta. \quad (3.3)$$

As these equations are the same as the minimal two-particle form factor equations in the sinh-Gordon theory, we borrow the results from there [21]

$$h^\gamma(\theta) = \mathcal{N}^\gamma \exp \left\{ 4 \int \frac{dt}{t} \frac{\sinh(tp^\gamma) \sinh(t(1-p^\gamma))}{\cosh(t) \sinh(2t)} \sin^2 \left(\frac{t}{\pi}(i\pi - \theta) \right) \right\} \quad (3.4)$$

where $\lambda^\gamma = \sin \pi p^\gamma$. The normalization

$$\mathcal{N}^\gamma = \exp \left\{ -2 \int \frac{dt}{t} \frac{\sinh(tp^\gamma) \sinh(t(1-p^\gamma))}{\cosh(t) \sinh(2t)} \right\} \quad (3.5)$$

is chosen such that the minimal form factor satisfies the following identity

$$h^\gamma(\theta + i\pi)h^\gamma(\theta) = \frac{\sinh \theta}{\sinh \theta + i\lambda^\gamma}. \quad (3.6)$$

Strictly speaking, this identification with the sinh-Gordon theory is valid only if $p^\gamma \in [0, 1]$, outside of this domain analytic continuation is needed.

Since the scattering matrix is trivial, $S \equiv 1$, and the reflection factor does not have any pole at $\frac{i\pi}{2}$, the Ansatz for the multiparticle form factor is

$$F_n^{\beta\alpha}(\theta_1, \theta_2, \dots, \theta_n) = \langle \mathcal{O}_{\beta\alpha} \rangle H_n Q_n(y_1, \dots, y_n) \prod_{i=1}^n r^{\beta\alpha}(\theta_i) \prod_{i < j} \frac{1}{y_i + y_j} \quad (3.7)$$

where Q is a symmetric polynomial. When the reflection factors are different the kinematical singularity axiom

$$-i \operatorname{Res}_{\theta=\theta'} F_{n+2}^{\beta\alpha}(-\theta+i\pi, \theta', \theta_1, \dots, \theta_n) = \left(R^\beta(\theta) - R^\alpha(\theta) \right) F_n^{\beta\alpha}(\theta_1, \dots, \theta_n) \quad (3.8)$$

recursively links Q_{n+2} to Q_n . Using that

$$r^{\beta\alpha}(-\theta+i\pi) r^{\beta\alpha}(\theta) = \frac{\sinh \theta}{\sinh \theta + i\lambda^\alpha} \frac{\sinh \theta}{\sinh \theta + i\lambda^\beta} \quad (3.9)$$

$$R^\beta(\theta) - R^\alpha(\theta) = \frac{2i \sinh \theta (\lambda^\alpha - \lambda^\beta)}{(\sinh \theta + i\lambda^\alpha)(\sinh \theta + i\lambda^\beta)} \quad (3.10)$$

and choosing $H_{2n} = (4(\lambda^\alpha - \lambda^\beta))^n$ we obtain a recursion, connecting either the even or the odd particle polynomials to each other, which reads as

$$Q_{n+2}(-y, y, y_1, \dots, y_n) = \prod_{i=1}^n (y + y_i)(-y + y_i) Q_n(y_1, \dots, y_n) \quad (3.11)$$

Let us choose $Q_0 = 1$, and solve the first few equations explicitly

$$Q_2(-y, y) = Q_0 \rightarrow Q_2 = 1 \quad (3.12)$$

$$Q_4(-y, y, y_1, y_2) = (y^2 - y_1^2)(y^2 - y_2^2) Q_2(y_1, y_2) \rightarrow Q_4 = \left((\sigma_2^{(4)})^2 + \sigma_1^{(4)} \sigma_3^{(4)} - 4\sigma_4^{(4)} \right)$$

where in the last line we introduced the elementary symmetric polynomials, defined as

$$\prod_{i=1}^n (y + y_i) = \sum_k y^{n-k} \sigma_k^{(n)}(y_1, \dots, y_n) \quad (3.13)$$

With this definition we have $\sigma_k^{(n)} = 0$ if $k < 0$ or $k > n$. In what follows we will usually omit the arguments of the symmetric polynomials, if it does not lead to any confusion. It is instructive to rewrite the solution by explicitly dividing by the product $\prod_{i<j} (y_i + y_j)$:

$$G_2 = \frac{Q_2}{y_{12}} = \frac{1}{y_{12}}, \quad G_4 = \frac{Q_4}{y_{12}y_{13}y_{14}y_{23}y_{24}y_{34}} = \frac{1}{y_{12}y_{34}} + \frac{1}{y_{13}y_{24}} + \frac{1}{y_{14}y_{23}} = \frac{1}{y_{34}} G_2 + \text{perm.} \quad (3.14)$$

where $y_{ij} = y_i + y_j$. This solution generalizes to

$$G_n = \frac{Q_n}{\prod_{i<j} y_{ij}} = \frac{1}{y_{nn-1}} G_{n-2} + \text{perm} = \sum_{\text{all pairings}} \frac{1}{\prod_{\text{pairs}(i,j)} y_{ij}} \quad (3.15)$$

Strictly speaking (3.15) gives the solution for even number of particles. However, similar calculation can be done for the odd particle sector starting from $Q_1 = 1$, and finally one arrives at the same formula (3.15), but in this case a pairing means that one of the y 's is left unpaired and does not contribute to the product. The resulting formula is very natural for a free theory and reflects Wick theorem. Actually it is not hard to see that this G_n solves the recursion equations since in the parametrization

$$F_n^{\beta\alpha}(\theta_1, \theta_2, \dots, \theta_n) = \langle \mathcal{O}_{\beta\alpha} \rangle H_n G_n(y_1, \dots, y_n) \prod_{i=1}^n r^{\beta\alpha}(\theta_i) \quad (3.16)$$

the kinematical recursion equation takes the form:

$$\lim_{y_{n+2} \rightarrow -y_{n+1}} y_{n+1n+2} G_{n+2}(y_1, \dots, y_n, y_{n+1}, y_{n+2}) = G_n(y_1, \dots, y_n) \quad (3.17)$$

which is satisfied by construction. In the following we try to directly solve the same model.

3.1.2 Direct solution of the model

The free massive scalar field $\Phi(x, t)$ restricted to the negative half-line $x \leq 0$ subject to the linear boundary condition

$$\partial_x \Phi(x, t)|_{x=0} = -\lambda m \Phi(0, t) \quad (3.18)$$

can be described by the following Lagrangian:

$$\mathcal{L} = \Theta(-x) \left(\frac{1}{2} (\partial_t \Phi)^2 - \frac{1}{2} (\partial_x \Phi)^2 - \frac{m^2}{2} \Phi^2 \right) - \delta(x) \frac{\lambda m}{2} \Phi^2 \quad (3.19)$$

This one parameter family of linear boundary conditions interpolates between Neumann $\partial_x \Phi|_{x=0} = 0$ (for $\lambda = 0$) and Dirichlet $\Phi|_{x=0} = 0$ (for $\lambda \rightarrow \infty$) boundary conditions and can be solved explicitly. The mode decomposition of the field is

$$\Phi(x, t) = \int_0^\infty \tilde{d}k \left\{ a(k) e^{-i\omega(k)t} \phi_k(x) + a^+(k) e^{i\omega(k)t} \phi_k^*(x) \right\}; \quad \phi_k(x) = e^{ikx} + R(k) e^{-ikx} \quad (3.20)$$

where $\tilde{d}k = \frac{dk}{4\pi\omega(k)}$ and creation/annihilation operators are normalized as

$$[a(k), a^+(k')] = 4\pi\omega(k) \delta(k - k') \quad , \quad k, k' > 0 \quad (3.21)$$

with $\omega(k) = \sqrt{m^2 + k^2}$, and the boundary condition fixes the reflection factor to be

$$R(k) = \frac{k - i\lambda m}{k + i\lambda m} \quad \longrightarrow \quad R(\theta) = \frac{\sinh \theta - i\lambda}{\sinh \theta + i\lambda} \quad (3.22)$$

The vacuum is defined as

$$a(k)|0\rangle = 0 \quad ; \quad k > 0 \quad (3.23)$$

and the states are created by acting successively with the creation operators $a^+(k)$. The wave functions are orthonormalized, satisfying

$$\int_{-\infty}^0 \phi_k(x) \phi_{k'}^*(x) dx = 2\pi \delta(k - k') \quad , \quad k, k' > 0 \quad ; \quad \phi_k^*(x) = R(-k) \phi_k(x) \quad (3.24)$$

and they also form a complete set

$$\int_0^\infty \frac{dk}{2\pi} \phi_k^*(x) \phi_k(y) = \delta(x - y) \quad (3.25)$$

These can be obtained by regularizing the integrals as

$$\int_{-\infty}^0 e^{ikx} dx = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^0 e^{i(k-i\epsilon)x} dx = \lim_{\epsilon \rightarrow 0} \frac{-i}{k - i\epsilon} = -i\mathbb{P}_{\frac{1}{k}} + \pi\delta(k) \quad (3.26)$$

We now turn to the problem of changing the boundary condition. Let us assume that for $t < 0$ the boundary condition has label λ^α , while for $t > 0$ it is changed to λ^β . The corresponding reflection factors are denoted by R^α and R^β , respectively. The expansion of the free field before and after the insertion of the boundary changing operator is

$$\Phi(x, t) = \begin{cases} \int_0^\infty \tilde{d}k \left\{ a_\alpha(k) e^{-i\omega(k)t} \phi_k^\alpha + a_\alpha^+(k) e^{i\omega(k)t} \phi_k^{\alpha*} \right\} & t < 0 \\ \int_0^\infty \tilde{d}k \left\{ a_\beta(k) e^{-i\omega(k)t} \phi_k^\beta + a_\beta^+(k) e^{i\omega(k)t} \phi_k^{\beta*} \right\} & t > 0 \end{cases} \quad (3.27)$$

As each set of modes form a complete system, we can expand each in terms of the other

$$\begin{aligned} A_{kk'}^{\alpha\beta} &\equiv \int_{-\infty}^0 \phi_k^{\alpha*}(x) \phi_{k'}^\beta(x) dx = \\ &= \frac{4k k' m (\lambda^\alpha - \lambda^\beta)}{(k^2 - k'^2)(k - im\lambda^\alpha)(k' + im\lambda^\beta)} + \frac{2\pi(m^2 \lambda^\beta \lambda^\alpha + k k')}{(k - im\lambda^\alpha)(k' + im\lambda^\beta)} \delta(k - k') \end{aligned} \quad (3.28)$$

where the first term is understood in the principal value sense, and $k, k' > 0$. The creation/annihilation operators can be related by demanding the continuity of the field $\Phi(x, t)$ and its momentum $\partial_t \Phi(x, t) = \Pi(x, t)$ at $t = 0$:

$$\begin{aligned} \omega(k) \Phi(x, 0) \pm i \Pi(x, 0) &= \\ &= \int_0^\infty \tilde{d}k' \left\{ a_\gamma(k') (\omega(k) \pm \omega(k')) + a_\gamma^+(k') R^\gamma(-k') (\omega(k) \mp \omega(k')) \right\} \phi_{k'}^\gamma(x) \end{aligned} \quad (3.29)$$

where γ can be either α or β . Comparing the two expressions we can extract that

$$a_\alpha(k) = \int_0^\infty \tilde{d}k' \left\{ a_\beta(k') (\omega(k) + \omega(k')) + R^\beta(-k') a_\beta^+(k') (\omega(k) - \omega(k')) \right\} A_{kk'}^{\alpha\beta} \quad (3.30)$$

An important effect of the boundary changing operator is that it changes the vacuum of the system: the vacuum for the α boundary condition, $a_\alpha(k)|0\rangle^\alpha = 0$, becomes a complicated excited state for the β boundary condition. As the transformation between the modes is linear we face with a Bogoliubov transformation, whose solution has an exponential form

$$\begin{aligned} |0\rangle^\alpha &= \mathcal{N}^{\alpha\beta} \left(1 + \int_0^\infty \tilde{d}k_0 K_1^{\alpha\beta}(k_0) a_\beta^+(k_0) \right) \times \\ &\quad \times \exp \left\{ \frac{1}{2} \iint_0^\infty \tilde{d}k_1 \tilde{d}k_2 K_2^{\alpha\beta}(k_1, k_2) a_\beta^+(k_1) a_\beta^+(k_2) \right\} |0\rangle^\beta \end{aligned} \quad (3.31)$$

where $K_1^{\alpha\beta}$ and $K_2^{\alpha\beta}$ are the solutions of

$$\int_0^\infty \tilde{d}k' (\omega(k) + \omega(k')) A_{kk'}^{\alpha\beta} K_1^{\alpha\beta}(k') = 0 \quad (3.32)$$

and

$$A_{kk'}^{\alpha\beta} R^\beta(-k') (\omega(k) - \omega(k')) + \int_0^\infty \tilde{d}k_1 A_{kk_1}^{\alpha\beta} (\omega(k) + \omega(k_1)) K_2^{\alpha\beta}(k_1, k') = 0. \quad (3.33)$$

The normalization is the overlap of the two vacua $\mathcal{N}^{\alpha\beta} = \mathcal{N}^{\beta\alpha*} = {}^\beta\langle 0|0\rangle^\alpha$. To see the validity of (3.31) one may check first that $a_\alpha(k)$ commutes with the factor in front of the

exponential provided (3.32) is satisfied. Then developing the exponential into Taylor series and acting with the β -representation of the a_α annihilation operator (3.30) it is not hard to see order-by-order that it annihilates the state. The equations (3.32,3.33) seem hard to solve, nevertheless one may check that the bootstrap solution satisfies them.

Comparing the bosonic algebra (3.21) to the free boson Zamolodchikov-Faddeev algebra (A.2) shows that they differ only in the normalization. We can thus relate the kernels $K_1^{\alpha\beta}$ and $K_2^{\alpha\beta}$ to the form factors, as

$$F_1^{\beta\alpha}(\theta) = \frac{1}{\sqrt{2}} {}^\beta \langle 0 | a_\alpha^+(k) | 0 \rangle^\alpha = \frac{1}{\sqrt{2}} \mathcal{N}^{\alpha\beta} K_1^{\beta\alpha*}(k) \quad (3.34)$$

and

$$F_2^{\beta\alpha}(\theta_1, \theta_2) = \frac{1}{2} {}^\beta \langle 0 | a_\alpha^+(k_1) a_\alpha^+(k_2) | 0 \rangle^\alpha = \frac{1}{2} \mathcal{N}^{\alpha\beta} K_2^{\beta\alpha*}(k_1, k_2) \quad (3.35)$$

with $k_i = m \sinh \theta_i$. Solving equations (3.32,3.33) thus would also determine the form factors. However, solving these equations is quite involved, we could not carry it out for the general case. In Appendix B we considered the case when we change the boundary condition from Neumann to Dirichlet. By mapping the problem to the already solved open-closed string vertex [3, 4], we managed to read off the solution which agrees with the bootstrap prediction.

3.2 The boundary scaling Lee-Yang model

The Lee-Yang theory is the simplest, non-unitary Conformal Field Theory, the $\mathcal{M}_{2,5}$ minimal model, with the central charge $c = -\frac{22}{5}$. The Virasoro algebra, Vir , has only two irreducible representation, denoted by V_0 and V_h with the highest weights 0 and $h = -\frac{1}{5}$. The periodic model carries the representation of two copies of the Virasoro algebra, $Vir \otimes \overline{Vir}$ and the modular invariance constrains the Hilbert space to decompose as

$$\mathcal{H} = V_0 \otimes \overline{V}_0 + V_h \otimes \overline{V}_h \quad (3.36)$$

We denote the corresponding primary fields by \mathbb{I} of the scaling dimension 0, and Φ of the scaling dimension $-\frac{2}{5}$, respectively.

A conformal boundary breaks the symmetry into a single Virasoro algebra. We will denote the two conformal boundary conditions by \mathbb{I} -boundary and Φ -boundary. The corresponding Hilbert spaces decompose as

$$\mathcal{H}_{\mathbb{I}} = V_0 \quad , \quad \mathcal{H}_{\Phi} = V_0 + V_h \quad (3.37)$$

There is only one primary field living on the \mathbb{I} -boundary, the identity field \mathbb{I} of weight 0, while on the Φ -boundary, beside the identity field, there is an other primary, ϕ , of the weight $-\frac{1}{5}$. There are nontrivial boundary fields of weight $-\frac{1}{5}$ interpolating the different boundary conditions, denoted by ψ and ψ^\dagger and the Hilbert space of the interpolating fields is $\mathcal{H}_\psi = \mathcal{H}_{\psi^\dagger} = V_h$.³

The boundary scaling Lee-Yang model is an integrable massive perturbation of the conformal boundary Lee-Yang model. It allows a boundary parameter [22]

$$S_\Phi(\lambda, \lambda_b) = S_\Phi + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^0 dx \phi(x, y) + \lambda_b \int_{-\infty}^{\infty} dy \varphi(y), \quad (3.38)$$

³The field ψ changes the boundary condition from ϕ to \mathbb{I} , while ψ^\dagger does the other way around.

where S_Φ denotes the action for the Lee-Yang model with the Φ -boundary condition imposed at $x = 0$, and λ, λ_b denote the bulk and boundary couplings, respectively. The action $S_{\mathbb{I}}(\lambda)$ of the perturbed theory with the identity boundary is similar, except the boundary perturbation is missing.

For $\lambda > 0$ the perturbed theory is a massive scattering theory having only a single particle type of mass $m(\lambda)$ with the S matrix [23]:

$$S(\theta) = \frac{\sinh \theta + i \sin \frac{\pi}{3}}{\sinh \theta - i \sin \frac{\pi}{3}} = - \left(\frac{1}{3} \right) \left(\frac{2}{3} \right) \quad (3.39)$$

where we used the block notation introduced in (2.27). The pole at $\theta = \frac{2\pi i}{3}$ indicates that the particle appears as a bound state of itself and such that the 3-particle coupling is $\Gamma = i\sqrt{2\sqrt{3}}$. The mass of the Lee-Yang particle as function of the perturbation parameter is given as

$$m(\lambda) = \kappa \lambda^{5/12} \quad , \quad \kappa = \frac{2^{19/5} \sqrt{\pi} \left(\Gamma(\frac{3}{5}) \Gamma(\frac{4}{5}) \right)^{5/12}}{5^{5/16} \Gamma(\frac{2}{3}) \Gamma(\frac{5}{6})}. \quad (3.40)$$

In the case of the \mathbb{I} boundary the reflection amplitude is

$$R^{\mathbb{I}}(\theta) = \left(\frac{1}{2} \right) \left(\frac{1}{6} \right) \left(-\frac{2}{3} \right) \quad (3.41)$$

which exhibits a pole at $i\frac{\pi}{2}$ with residue $g_{\mathbb{I}} = -2i\sqrt{(2\sqrt{3}-3)}$. This shows that the \mathbb{I} boundary can emit a virtual particle with zero energy but there are no bound-states on this boundary.

The reflection factor of the Φ -boundary depends on the strength of the boundary coupling constant λ_b as [22]

$$R^\Phi(\theta) = R^{\mathbb{I}}(\theta) R_\phi(\theta) \quad , \quad R_\phi(\theta) = S(\theta - \theta_0) S(\theta + \theta_0) \quad , \quad \theta_0 = i\pi \frac{3-b}{6}, \quad (3.42)$$

where the dimensionless parameter b is related to the dimensionful λ_b as

$$\lambda_b(b) = \sin \left(\left(b + \frac{1}{2} \right) \frac{\pi}{5} \right) m(\lambda)^{6/5} \lambda_{crit} \quad , \quad \lambda_{crit} = -\pi^{3/5} 2^{4/5} 5^{1/4} \frac{\sin \frac{2\pi}{5}}{\sqrt{\Gamma(\frac{3}{5}) \Gamma(\frac{4}{5})}} \left(\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})} \right)^{6/5}. \quad (3.43)$$

The fundamental range of the parameter b is $[-3, 2]$ and we have no boundary bound-state only in the region $b \in [-3, -1]$. This boundary reflection factor can also emit a virtual zero energy particle with amplitude

$$g_\Phi(b) = \frac{\cosh \theta_0 + \sin \frac{\pi}{3}}{\cosh \theta_0 - \sin \frac{\pi}{3}} g_{\mathbb{I}} \quad (3.44)$$

Note that $R^{\mathbb{I}}(\theta)$ is identical to $R^\Phi(\theta)$ at $b = 0$ and so both have a pole at $\theta = \frac{i\pi}{2}$ coming from the $(\frac{1}{2})$ block, but their g factors differ in a sign [24]. We also note that the Φ -boundary can be obtained by placing an integrable defect with transmission factor $T(\theta)$ in front of the identity boundary

$$R^\Phi(\theta) = T_-(\theta) R^{\mathbb{I}}(\theta) T_+(\theta) \quad (3.45)$$

In particular, the transmission factor satisfies $T_{\mp}(\theta) = S(\theta \mp \theta_0)$, thus it can be interpreted as an imaginary momentum bound particle. This, however, does not mean that the Φ -boundary is a boundary bound-state as the \mathbb{I} boundary has no bound-states.

In the following we consider the situation in which we have the analogue of $S_{\Phi}(\lambda, \lambda_b)$ for $t < 0$ and $S_{\mathbb{I}}(\lambda)$ for $t > 0$, (or the other way around). The change in the boundary condition is implemented by inserting the off-critical versions of ψ or ψ^{\dagger} or their descendants and we analyze the form factors of these operators. We use the general parametrization

$$F_n^{\mathcal{O}_{\beta\alpha}}(\theta_1, \theta_2, \dots, \theta_n) = \langle \mathcal{O}_{\beta\alpha} \rangle H_n^{\beta\alpha} \prod_{i=1}^n \frac{r^{\beta\alpha}(\theta_i)}{y_i} \prod_{i < j} \frac{f(\theta_i - \theta_j) f(\theta_i + \theta_j)}{(y_i + y_j)} Q_n^{\mathcal{O}_{\beta\alpha}}(y_1, y_2, \dots, y_n). \quad (3.46)$$

where we explicitly spelled out which quantities depend only on the various boundary conditions and which depend on the operator itself. From now on we will omit the operator if it does not lead to any confusion.

The minimal bulk two particle form factor, which has only a single zero at $\theta = 0$ and a pole at $\theta = \frac{2\pi i}{3}$ in the strip $0 \leq \Im m(\theta) < \pi$, has the form [25]:

$$f(\theta) = \frac{y-2}{y+1} v(i\pi - \theta) v(-i\pi + \theta) \quad , \quad y = e^{\theta} + e^{-\theta} \quad (3.47)$$

where

$$v(\theta) = \exp \left\{ 2 \int_0^{\infty} \frac{dt}{t} e^{i\frac{\theta t}{\pi}} \frac{\sinh \frac{t}{2} \sinh \frac{t}{3} \sinh \frac{t}{6}}{\sinh^2 t} \right\}. \quad (3.48)$$

It satisfies the important identities

$$f(\theta) f(\theta + i\pi) = \frac{\sinh \theta}{\sinh \theta - i \sin \frac{\pi}{3}} \quad , \quad \frac{f(\theta + \frac{i\pi}{3}) f(\theta - \frac{i\pi}{3})}{f(\theta)} = \frac{\cosh \theta + 1/2}{\cosh \theta + 1}. \quad (3.49)$$

The one-particle minimal boundary changing form factor is parametrized as

$$r^{\mathbb{I}\Phi}(\theta) = h^{\Phi}(\theta) h^{\mathbb{I}}(i\pi - \theta) = r^{\mathbb{I}\mathbb{I}}(\theta) r_{\phi}(\theta) \quad (3.50)$$

where

$$r^{\mathbb{I}\mathbb{I}}(\theta) = 4i \sinh \theta \exp \left\{ \int_0^{\infty} \frac{dt}{t} \frac{\sinh(t) - \cosh\left(\frac{it}{2} - \frac{\theta t}{\pi}\right) \left(\sinh \frac{5t}{6} + \sinh \frac{t}{2} - \sinh \frac{t}{3}\right)}{\sinh \frac{t}{2} \sinh t} \right\} \quad (3.51)$$

is the minimal form factor for the identity boundary condition. This representation is valid on the strip $0 \leq \Im m(\theta) \leq \pi$ and can be extended by analytic continuation outside this region. The identity boundary reflection factor satisfies

$$\begin{aligned} r^{\mathbb{I}\mathbb{I}}(i\pi + \theta) r^{\mathbb{I}\mathbb{I}}(\theta) f(i\pi - 2\theta) &= y^2(y^2 - 4) \\ \frac{r^{\mathbb{I}\mathbb{I}}(\theta + \frac{i\pi}{3}) r^{\mathbb{I}\mathbb{I}}(\theta - \frac{i\pi}{3})}{r^{\mathbb{I}\mathbb{I}}(\theta)} f(2\theta) &= y^2 - 3 \\ \frac{r^{\mathbb{I}\mathbb{I}}(\frac{i\pi}{2})}{v(0)} &= 4(\sqrt{3} - 3) \end{aligned} \quad (3.52)$$

From the parametrization (3.50) follows that r_ϕ satisfies

$$r_\phi(\theta) = R_\phi(\theta)r_\phi(-\theta) \quad , \quad r_\phi(i\pi - \theta) = r_\phi(i\pi + \theta) \quad (3.53)$$

and has no zeros or poles in the physical strip, thus the Karowski-Weisz theorem implies

$$r_\phi(\theta) = \mathcal{N} \exp \left\{ 2 \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{b+1}{6}t + \sinh \frac{b-1}{6}t - \sinh \frac{b+7}{6}t - \sinh \frac{b+5}{6}t}{\sinh^2 t} \sin^2 \left(\frac{i\pi - \theta}{2\pi} t \right) \right\}. \quad (3.54)$$

This representation of r_ϕ is valid only for $b \in [-3, -1]$ and $0 \leq \Im(\theta) \leq 2\pi$, and defined by analytic continuation outside this domain. If chose the normalization

$$\mathcal{N} = -\frac{1}{4} \exp \left\{ 2 \int_0^\infty \frac{dt}{t} \frac{\cosh \left(\frac{b+3}{6}t \right) \left[\sinh \frac{t}{3} + \sinh \frac{2t}{3} \right] - \sinh(t)}{\sinh^2(t)} \right\} \quad (3.55)$$

then $r_\phi(\theta)$ satisfies

$$r_\phi(\theta)r_\phi(\theta + i\pi) = \frac{1}{(y_0 - y_-)(y_0 + y_+)} \quad (3.56)$$

$$\frac{r_\phi(\theta + \frac{i\pi}{3})r_\phi(\theta - \frac{i\pi}{3})}{r_\phi(\theta)} = \frac{1}{y + y_0} \quad (3.57)$$

$$r_\phi\left(\frac{i\pi}{2}\right) = \frac{1}{y_0 - \sqrt{3}} \quad (3.58)$$

with

$$y_+ = \omega e^\theta + \omega^{-1} e^{-\theta} \quad , \quad y_- = \omega e^{-\theta} + \omega^{-1} e^\theta \quad , \quad y_0 = 2 \cosh \theta_0 \quad , \quad \omega = e^{i\frac{\pi}{3}}. \quad (3.59)$$

The minimal one particle boundary changing form factor corresponding to $\alpha = \mathbb{I}$ and $\beta = \Phi$ is given as

$$r^{\Phi\mathbb{I}}(\theta) = -h^{\mathbb{I}}(\theta)h^\Phi(i\pi - \theta) = -r^{\mathbb{III}}(\theta)r_\phi(i\pi - \theta) \quad (3.60)$$

where we defined an extra sign into $r^{\Phi\mathbb{I}}$ for later convenience.

By choosing the normalization $H_n^{\mathbb{I}\Phi} = H_n^{\Phi\mathbb{I}} = \left(\frac{i\sqrt[4]{3}}{v(0)\sqrt{2}} \right)^n$ the recursion relations for the polynomials become

$$Q_{n+2}^{\beta\alpha}(y_+, y_-, y_1, \dots, y_n) = D_n^{\beta\alpha}(y|y_1, \dots, y_n) Q_{n+1}^{\beta\alpha}(y, y_1, \dots, y_n) \quad (3.61)$$

$$Q_{n+2}^{\beta\alpha}(y, -y, y_1, \dots, y_n) = P_n^{\beta\alpha}(y|y_1, \dots, y_n) Q_n^{\beta\alpha}(y_1, \dots, y_n) \quad (3.62)$$

$$Q_{n+1}^{\beta\alpha}(0, y_1, \dots, y_n) = B_n^{\beta\alpha}(y_1, \dots, y_n) Q_n^{\beta\alpha}(y_1, \dots, y_n) \quad (3.63)$$

with

$$P_n^{\mathbb{I}\Phi}(y|y_1, \dots, y_n) = P_{n+1}^{\mathbb{III}}(y|y_0, y_1, \dots, y_n), \quad P_n^{\Phi\mathbb{I}}(y|y_1, \dots, y_n) = P_{n+1}^{\mathbb{III}}(y|-y_0, y_1, \dots, y_n) \quad (3.64)$$

$$D_n^{\mathbb{I}\Phi}(y|y_1, \dots, y_n) = D_{n+1}^{\mathbb{III}}(y|y_0, y_1, \dots, y_n), \quad D_n^{\Phi\mathbb{I}}(y|y_1, \dots, y_n) = D_{n+1}^{\mathbb{III}}(y|-y_0, y_1, \dots, y_n) \quad (3.65)$$

$$B_n^{\mathbb{I}\Phi}(y_1, \dots, y_n) = B_{n+1}^{\mathbb{III}}(y_0, y_1, \dots, y_n), \quad B_n^{\Phi\mathbb{I}}(y_1, \dots, y_n) = B_{n+1}^{\mathbb{III}}(-y_0, y_1, \dots, y_n) \quad (3.66)$$

and

$$D_n^{\text{III}}(y|y_1, \dots, y_n) = \prod_{i=1}^n (y + y_i) \quad (3.67)$$

$$P_n^{\text{III}}(y|y_1, \dots, y_n) = \frac{\prod_{i=1}^n (y_i - y_-)(y_i + y_+) - \prod_{i=1}^n (y_i + y_-)(y_i - y_+)}{2(y_+ - y_-)} \quad (3.68)$$

$$B_n^{\text{III}}(y_1, \dots, y_n) = \frac{\prod_{i=1}^n (y_i + \sqrt{3}) - \prod_{i=1}^n (y_i - \sqrt{3})}{2\sqrt{3}} \quad (3.69)$$

3.2.1 Form factors of the primary boundary changing fields

The form factor recurrence relations (3.61,3.62,3.63) have many sets of solution. We expect that the ones with the mildest ultraviolet behaviour correspond to the off-critical versions of the primary boundary changing fields, ψ and ψ^\dagger , with the appropriate boundaries. Observe that the recursion relations are exactly the same that we would get for $Q_{n+1}^{\text{III}}(\pm y_0, y_1, \dots, y_n)$ for the boundary form factors on the II boundary condition [12]. So that one may expect to get the Q -polynomials of the fields ψ and ψ^\dagger from the polynomials corresponding to the off-critical version of the energy-momentum tensor with identity boundary, which is the lowest-lying solution in the that case, by setting $y_{n+1} \rightarrow \pm y_0$. However, there is an essential difference between the case of boundary changing operators and the identity boundary case, namely in the latter case the one-particle form factor does not have the boundary kinematical pole while in the former case it does. The vanishing of the residue of the boundary kinematical pole requires $Q_1^{\text{III}}(0) = 0$ thus $Q_0^{\text{III}} = 0$ for all operators living on the identity boundary, but we expect $Q_0^\psi = Q_0^{\psi^\dagger} = 1$.

The solution for the off-critical energy-momentum tensor with identity boundary condition was determined in [26] and reads as

$$Q_1^T = \sigma_1^{(1)} \quad ; \quad Q_2^T = \sigma_1^{(2)} \quad ; \quad Q_3^T = \left(\sigma_1^{(3)}\right)^2 \quad ; \quad Q_n^T = \left(\sigma_1^{(n)}\right)^2 \det \Xi^{(n)} \quad (3.70)$$

for $n \geq 4$ where the $(n-3) \times (n-3)$ matrix function is defined as

$$\Xi_{ij}^{(n)} = \sum_{k \in \mathbb{Z}} 3^k \binom{i-j+k}{k} \sigma_{3j-2i+1-2k}^{(n)} \quad , \quad 1 \leq i, j \leq n-3 \quad (3.71)$$

However it is still possible to generate the form factor solutions for the boundary changing primaries from the solution for the energy-momentum tensor. Let us observe that the $\sigma_1^{(n)}$ symmetric polynomial, introduced in (3.13), is a zero mode of the recurrence equations (3.61,3.62,3.63), i.e.

$$\begin{aligned} \sigma_1(y_+, y_-, y_1 \dots y_n) &= \sigma_1(y, y_1, \dots, y_n) \quad , \quad \sigma_1(-y, y, y_1, \dots, y_n) = \sigma_1(y_1, \dots, y_n) \\ \sigma_1(0, y_1, \dots, y_n) &= \sigma_1(y_1, \dots, y_n) \end{aligned} \quad (3.72)$$

thus every solution can be multiplied or, if divisible, divided by σ_1 ! Dividing the Q_{n+1}^T polynomial, corresponding to the energy-momentum tensor in the identity boundary case, by

$\sigma_1^{(n+1)}$ and evaluating it at $y_{n+1} = \pm y_0$ will generate the solution for ψ and ψ^\dagger with the appropriate initial conditions, $Q_0^\psi = Q_0^{\psi^\dagger} = 1$,

$$Q_n^\psi(y_1, \dots, y_n) = \frac{Q_{n+1}^T}{\sigma_1^{(n+1)}} \Big|_{(y_0, y_1, \dots, y_n)}, \quad Q_n^{\psi^\dagger}(y_1, \dots, y_n) = \frac{Q_{n+1}^T}{\sigma_{n+1}^{(1)}} \Big|_{(-y_0, y_1, \dots, y_n)} \quad (3.73)$$

3.2.2 Two point functions of boundary operators and their UV limits

Let us consider the off-critical two-point functions of the Euclidean version of the model

$$\langle \varphi_1(r) \varphi_2(0) \rangle \quad (3.74)$$

where φ_i ($i = 1, 2$) is one of the off-critical version of the boundary fields ϕ , ψ and ψ^\dagger compatible with the corresponding boundary conditions. The two point function can be computed via its spectral representation

$$\langle 0 | \varphi_1(r) \varphi_2(0) | 0 \rangle = \sum_{n=0}^{\infty} \int_{\theta_1 > \dots > \theta_n > 0} \frac{d\theta_1}{2\pi} \dots \frac{d\theta_n}{2\pi} e^{-r \Delta E_{\text{bdry}}^{\varphi_1} - m r \sum_i \cosh \theta_i} F_n^{\varphi_1} F_n^{\varphi_2+} \quad (3.75)$$

where

$$\begin{aligned} F_n^{\varphi_1} &= \langle 0 | \varphi_1(0) | \theta_1, \dots, \theta_n \rangle_{in} = F_n^{\varphi_1}(\theta_1, \dots, \theta_n) \\ F_n^{\varphi_2+} &= {}_{in} \langle \theta_1, \dots, \theta_n | \varphi_2(0) | 0 \rangle = F_n^{\varphi_2}(\pi + \theta_1, \dots, \pi + \theta_n) \end{aligned} \quad (3.76)$$

and $\Delta E_{\text{bdry}}^{\varphi_1}$ is the difference of the boundary energies of the boundary conditions in between φ_1 interpolates,

$$\Delta E_{\text{bdry}}^\phi = 0 \quad , \quad \Delta E_{\text{bdry}}^{\psi^\dagger} = -\Delta E_{\text{bdry}}^\psi = \frac{y_0}{2} \quad (3.77)$$

Truncation of the series (3.75) up to two particle term gives a good approximation even for small separation which can be compared to the CFT prediction. Assuming that there is a one-to-one correspondence between the field content of the perturbed theory and the CFT (apart from some additive renormalization constant [25]) we can use the operator product expansion of the CFT

$$\varphi_1(r) \varphi_2(0) \sim \sum_j \frac{C_{12}^j \varphi_j}{|r|^{h_1+h_2-h_j}} \quad (3.78)$$

where the sum runs over all the boundary fields, and h_j denotes the weights of the fields. Choosing φ_1 and φ_2 to be primaries and keeping the leading contributions in (3.78) with the lowest weights, i.e. the primaries appearing in the OPE of φ_1 and φ_2 , we get a good approximation of the short distance behavior of the two point functions. The OPEs of interest are

$$\phi(z) \psi^\dagger(w) = C_{\phi\psi^\dagger}^{\psi^\dagger} |z-w|^{1/5} \psi^\dagger(w) + \dots \quad ; \quad \psi(z) \phi(w) = C_{\psi\phi}^\psi |z-w|^{1/5} \psi(w) + \dots \quad (3.79)$$

with the structure constants

$$C_{\phi\psi^\dagger}^{\psi^\dagger} = C_{\psi\phi}^\psi = -\sqrt{\frac{2}{1+\sqrt{5}}} \sqrt{\frac{\Gamma(\frac{1}{5}) \Gamma(\frac{6}{5})}{\Gamma(\frac{3}{5}) \Gamma(\frac{4}{5})}} \quad (3.80)$$

As the exact vacuum expectation values of the boundary (changing) fields are only known for ϕ [27]

$$\langle \phi \rangle = -\frac{5}{6|\lambda_{crit}|} \frac{\cos(b\pi/6)}{\cos(\pi(b+1/2)/5)} m^{-\frac{1}{5}} \quad (3.81)$$

we will consider the normalized two point functions⁴

$$\frac{\langle \psi(r)\phi(0) \rangle}{\langle \psi \rangle \langle \phi \rangle} = \frac{C_{\psi\phi}^\psi}{\langle \phi \rangle} (mr)^{1/5} + \dots \quad ; \quad \frac{\langle \phi(r)\psi^\dagger(0) \rangle}{\langle \phi \rangle \langle \psi^\dagger \rangle} = \frac{C_{\phi\psi^\dagger}^{\psi^\dagger}}{\langle \phi \rangle} (mr)^{1/5} + \dots \quad (3.82)$$

As the form factors are also proportional to the vacuum expectation values of the fields, it drops out in the normalized version.

For the numerical implementation of the truncated form factor series we need the form factors of the boundary field ϕ . They are parametrized as

$$F_n^\phi(\theta_1, \dots, \theta_n) = \langle \phi \rangle H_n^{\Phi\Phi} \prod_{i=1}^n \frac{r^{\Phi\Phi}(\theta_i)}{y_i} \prod_{i<j} \frac{f(\theta_i - \theta_j) f(\theta_i + \theta_j)}{y_i + y_j} Q_n^\phi(y_1, \dots, y_n) \quad (3.83)$$

with

$$r^{\Phi\Phi}(\theta) = \frac{1}{4(\sinh \theta - i \sin \pi \frac{b-1}{6})(\sinh \theta - i \sin \pi \frac{b+1}{6})} r^{\text{III}}(\theta) \quad (3.84)$$

and $H_n^{\Phi\Phi} = \left(\frac{i\sqrt[4]{3}}{v(0)\sqrt{2}} \right)^n$. The polynomials Q_n^ϕ are calculated explicitly in [26], we only need the first few of them, which are

$$Q_1^\phi = \sigma_1^{(1)} \quad , \quad Q_2^\phi = \sigma_1^{(2)} (\sigma_2^{(2)} + 3 - y_0^2) . \quad (3.85)$$

We numerically calculated the one- and two-particle contributions to the normalized two point functions and plotted against the CFT prediction, shown in Figure 2, which shows a good agreement. This is a solid confirmation of our solutions for the form factors of ψ and ψ^\dagger .

3.2.3 Classification of the form factor solutions

In this subsection we classify the polynomial solutions of the recursion relations following [28, 29, 30, 14]. The asymptotic degree of a form factor solution is defined as

$$\lim_{\Lambda \rightarrow \infty} F_n^{\beta\alpha}(\theta_1 + \Lambda, \dots, \theta_n + \Lambda) = e^{x_n \Lambda} + \dots \quad (3.86)$$

and is in one-to-one correspondence with the UV scaling dimension of the operator. Using the parametrization of the form factors together with their asymptotic behaviour their degree turns out to be

$$x_n = \deg Q_n - \frac{n(n-1)}{2} \quad (3.87)$$

The form factor of each boundary changing operator starts at a given particle number and all form factors with more particles are uniquely determined from this first. Such family of

⁴Here the ground state expectation value is meant as the matrix element between the lowest energy states corresponding to the various boundary condition. It can either be the the highest weight state of the V_0 module, $|0\rangle$, in the identity boundary case or the highest weight state of the V_h module, $|\phi\rangle$, in the Φ -boundary case.

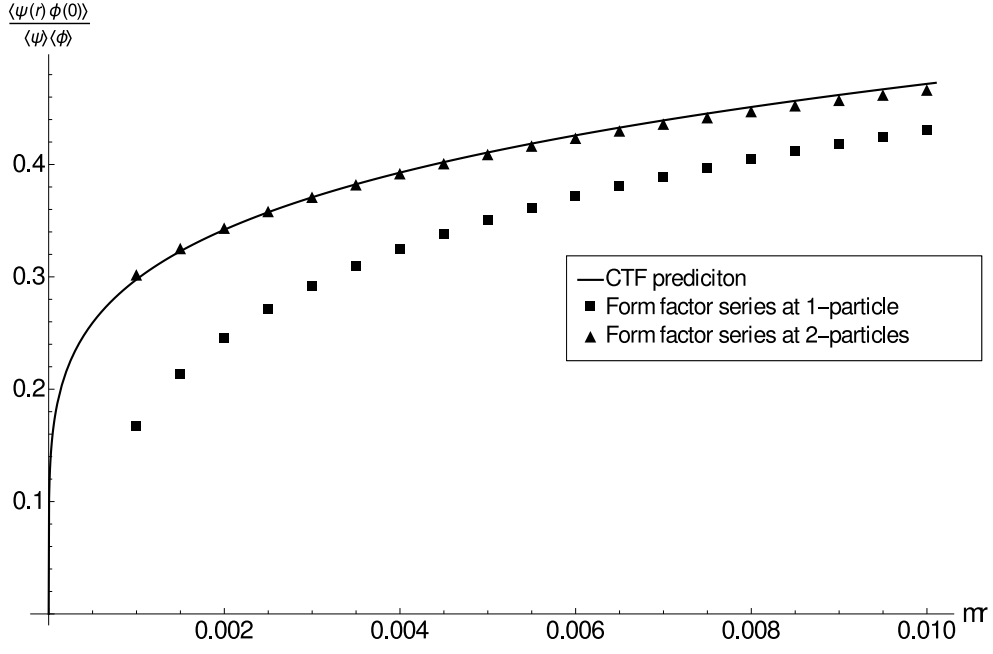


Figure 1: The normalized $\langle \psi(r) \phi(0) \rangle$ two point function at $b = -2$.

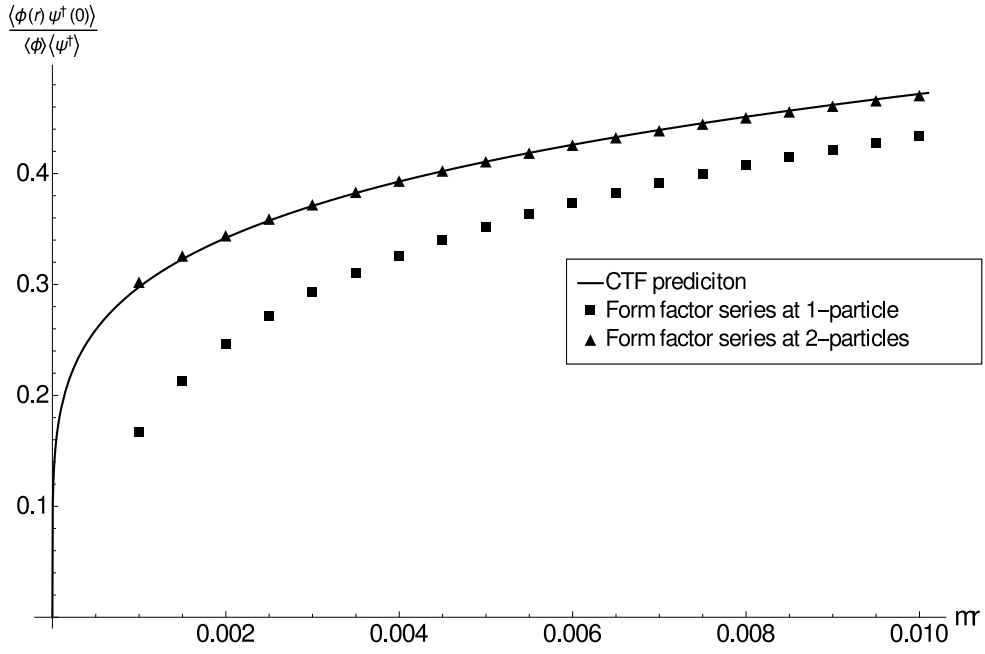


Figure 2: The normalized $\langle \phi(r) \psi^\dagger(0) \rangle$ two point function at $b = -2$.

solutions, which defines the operator, is called the form factor tower and the scaling dimension can be read off from the degree of the top of the tower. So far we considered only the solution which starts at the first level and has the mildest asymptotic growth, but there are also other solutions. They correspond to the so-called kernel solutions and can start at any higher level. An n^{th} level kernel solution is defined as a polynomial of n variables whose value is zero at the positions of all the singularity axioms. In the case of the boundary Lee-Yang model they are given as

$$Q_n = \sigma_{k_1}^{(n)} \dots \sigma_{k_l}^{(n)} K_n \quad ; \quad K_n = \prod_{1 \leq i < j \leq n} (y_i + y_j) \prod_{1 \leq i < j \leq n} (y_i^2 + y_i y_j + y_j^2 - 3) \prod_{i=1}^n y_i \quad (3.88)$$

where $0 < k_1 \leq k_2 \leq \dots \leq k_l \leq n$. The corresponding form factor has degree

$$x_n = k_1 + \dots + k_l + n^2 \quad (3.89)$$

Formally we can consider the fundamental solution corresponding to $K_1 = 1$. Its descendant $\sigma_1^n K_1$ is nothing but its n^{th} derivative. The generating function of all the solutions is

$$1 + \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} P(m|n) q^{l+n^2} \quad (3.90)$$

where $P(m|n)$ denotes the number of partitions of the number m such that none of summands is greater than n , and the extra 1 corresponds to K_1 . Using

$$\sum_{m=0}^{\infty} P(m|n) q^m = \prod_{i=1}^n (1 - q^i)^{-1} \quad (3.91)$$

and the Rogers-Ramanujan identity we can write

$$1 + \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} P(l|n) q^{l+n^2} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{l=1}^n (1 - q^l)} = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})} = \tilde{\chi}_{-\frac{1}{5}} \quad (3.92)$$

which is the truncated character of the $h = -\frac{1}{5}$ representation. Thus at each level we found exactly the same number of form factor solutions as many state exist at that conformal level. As there is an isomorphism between states and local boundary changing operators in a CFT we can see that there is a one-to-one correspondence between the form factor solutions and local boundary changing operators.

4 Conclusion

In this paper we established the form factor bootstrap program for boundary condition changing operators in integrable models. Our proposal fills some gap as, although the complete set of form factor axioms were known for a long time for bulk [31, 32], boundary [12] and defect [33] models, and also for some non-local operator insertions [34], the complete set of axioms for the form factors of local boundary changing operators were missing. We have tested the consistency of the form factor axioms and presented the general procedure to determine their solutions.

The first step of the method is the calculation of the one-particle minimal form factor. Whenever the reflection factors of the two boundaries can be written as a product of blocks (2.27), the ingredients of the minimal solution are granted by the theorem of Karowski and Weisz [20]. Then, a general multiparticle form factor can be parametrized in terms of the minimal boundary form factor and the bulk two-particle minimal form factor, which automatically satisfies some of the axioms. This parametrization includes a polynomial factor, and the rest of the form factor axioms give restrictive recursive relations connecting these polynomials. There is a one-to-one correspondence between the families of solutions of the recurrence relations and the operator content of the model [28, 29, 30, 14].

In the pioneering paper [9] the authors analyzed in detail the free massive fermion and the sinh-Gordon model. Here, we analyzed two other models in detail. First, in the boundary condition changing free boson theory, we solved the form factor bootstrap axioms. If, at a moment, the boundary condition is changed, the vacuum of the pre-quench system becomes an excited state of the post-quench one. We presented the explicit relation of the two vacua involving two kernel functions satisfying specific integral equations. We gave the relation of these kernel functions to the one- and two-particle form factors. When the boundary condition is changed from Neumann to Dirichlet, we showed that the form factor bootstrap solutions indeed satisfy these integral equations. It would be interesting to prove that it also holds for the generic case.

A finite volume analysis was presented in the case when the boundary condition is switched from Neumann to Dirichlet, by introducing a second boundary at $x = -L$ with Neumann boundary condition. In fact, the boundary condition of the new boundary is not relevant as we take the $L \rightarrow \infty$ limit at the end. The before and after quench boson creation and annihilation operators, as in the infinite volume case, are related to each other by a Bogoliubov-type transformation. By hermitian conjugation we can flip back the outgoing Dirichlet states and the new incoming states are now tensor products of two free boson states. The vertex state is defined such that the overlap of an incoming and an outgoing state before the flipping, i.e. the form factor of the quench operator, is equal to the overlap of the flipped incoming state and the vertex state. We parametrized the vertex state in terms of the so-called Neumann coefficients, and the relations connecting the creation and annihilation operators result restrictive equations for the Neumann coefficients. A similar problem had been analyzed in the context of the open-closed string vertex [3, 4]. If we consider Dirichlet boundary condition on the open string than the resulting equations for the string vertex can be mapped to our equations for the Neumann coefficients, thus we could simply read of the solutions. By definition, the vertex state contain all the information of the form factors, thus by taking the $L \rightarrow \infty$ limit of the Neumann coefficients we could determine directly the infinite volume form factors of the boundary changing operator. The resulting functions coincide with the bootstrap prediction which confirms the validity of our axioms.

We also considered the scaling Lee-Yang model. There are only two integrable boundary condition, the identity boundary and the so-called Φ -boundary. We studied both the case when we switch from the identity to the Φ -boundary and the other way around. First, we calculated the minimal boundary-changing one-particle form factors and then we derived the recursive relations for the polynomials appearing in the parametrization of the multiparticle form factors. These recurrence equations turned out to be very similar to the ones for the (unquenched) identity boundary [12], whose solutions are known [26]. We gave the explicit solutions for the form factors corresponding to the boundary changing operators with the mildest ultraviolet behaviour, i.e. the off-critical versions of the conformal boundary changing

primary fields. By analyzing the structure of the recurrence relations, we found their common kernels. By counting the kernel solutions we showed that there is a one-to-one correspondence between the operator content of the theory and the towers of solutions of the form factor axioms. Finally, we studied the two-point correlation functions of a boundary and a boundary changing operator. Their spectral series, truncated at two-particle level, give a good approximation of the two-point function even in relatively small volume. We compared this against the conformal field theory prediction, and we found a good agreement. This supports the validity of our form factor solutions.

In the future it would be interesting to generalize the truncated conformal space approach to describe boundary changing operators in order to test our results, similarly how this check was carried out for boundary form factors in [18] and for defect form factors in [35].

Our framework is very general and can be directly used to calculate the form factors of the boundary changing operators in other diagonal models. The generalization of the program for non-diagonal theories is also very interesting.

From the quench problem point of view our result provides the exact overlap of the pre-quench vacuum with all the post-quench states. This result could be used to calculate interesting physical quantities like correlation functions which can shed light on thermalization or can characterize steady states.

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A Formal derivation of the axioms from the ZF algebra

Here we present a formal derivation of our axioms from the Zamolodchikov-Faddeev algebra⁵. This algebra contains the exact operators $Z^+(\theta)$ and $Z(\theta)$ which create and annihilate particles. Formally they can be continued for complex rapidities and the crossing transformation relates them as

$$Z(\theta) = Z^+(\theta + i\pi) \quad (\text{A.1})$$

These operators satisfy an exchange axiom including the exact scattering matrix

$$Z^+(\theta_1)Z^+(\theta_2) = S(\theta_1 - \theta_2)Z^+(\theta_2)Z^+(\theta_1) + 2\pi\delta(\theta_1 - \theta_2 - i\pi) \quad (\text{A.2})$$

such that the exchange of the creation and annihilation operators contains the δ function, too.

In the presence of the boundary we introduce the boundary operators:

$$|0\rangle^\alpha = B_\alpha^+|0\rangle \quad , \quad {}^\beta\langle 0| = \langle 0|B_\beta \quad (\text{A.3})$$

⁵Similar consideration had been presented in [9].

such that

$$Z^+(\theta)B_\alpha^+ = R_\alpha(\theta)Z^+(-\theta)B_\alpha^+ + 2\pi\delta(\theta - \frac{i\pi}{2})\frac{g_\alpha}{2}B_\alpha^+ \quad (\text{A.4})$$

and

$$B_\beta Z(\theta) = B_\beta Z(-\theta)R_\beta(-\theta) + 2\pi\delta(\theta + \frac{i\pi}{2})\frac{g_\beta}{2}B_\beta \quad (\text{A.5})$$

The form factor axioms can be derived from the representation

$$F_n^{\mathcal{O}_{\beta\alpha}}(\theta_1, \dots, \theta_n) = \langle 0 | B_\beta \mathcal{O}_{\beta\alpha}(0) Z^+(\theta_1) \dots Z^+(\theta_n) B_\alpha^+ | 0 \rangle \quad (\text{A.6})$$

by assuming

$$[\mathcal{O}_{\beta\alpha}(0), Z^+(\theta)] = 0 \quad (\text{A.7})$$

B Changing the boundary condition from Neumann to Dirichlet

In this Appendix we analyze a simplified situation in which the Neumann boundary condition is changed to Dirichlet in the free boson theory. As a start we recall the bootstrap solution of the problem and show how it solves explicitly the constraints coming from the direct quantization. In the direct quantization the creation and annihilation operators of the two boundary conditions are related to each other by an infinite dimensional linear transformation. As a consequence, the vacuum state of the Neumann boundary condition is a complicated coherent state for the Dirichlet boundary (3.31), and the appearing kernels, the solutions of (3.32,3.33), can be found by inverting an infinite dimensional matrix, $A_{kk'}$. Although we cannot invert this matrix, we can show that the bootstrap solution provides a solution for the kernels.

In order to find the solution directly we put the system into a finite volume by introducing Neumann condition at the other end. We can map this finite volume problem to the open closed string vertex problem [3, 4] and the adopted solution in the infinite volume limit indeed reproduces the bootstrap result.

B.1 Bootstrap solution

Let us specify the bootstrap solution of Section 3.1 for the case when the Neumann boundary condition, labeled by $\alpha = +$ with reflection factor $R^\alpha(\theta) \equiv 1$, is changed to the Dirichlet boundary, labeled by $\beta = -$ with reflection factor $R^\beta(\theta) \equiv -1$. This limiting case can be obtained from the general considerations as the $\lambda^\alpha \rightarrow 0$ and $\lambda^\beta \rightarrow \infty$ limits. First, we need to calculate the one particle minimal form factor

$$r^{-+}(\theta) = h^+(\theta)h^-(i\pi - \theta) \quad (\text{B.1})$$

which turns out to be

$$h^+(\theta) = 1 \quad , \quad h^-(\theta) = 2 \sinh \frac{\theta}{2} \quad , \quad r^{-+}(\theta) = 2 \sinh \left(\frac{i\pi - \theta}{2} \right). \quad (\text{B.2})$$

We choose the normalization such that

$$r^{-+}(\theta)r^{-+}(i\pi + \theta) = -2i \sinh \theta. \quad (\text{B.3})$$

The general n -particle form factor is parametrized as

$$F_n^{-+}(\theta_1, \theta_2, \dots, \theta_n) = \mathcal{N} H_n G_n(y_1, \dots, y_n) \prod_{i=1}^n r^{-+}(\theta_i) \quad ; \quad y_i = e^{\theta_i} + e^{-\theta_i} \quad (\text{B.4})$$

where $\mathcal{N} = -\langle 0|0 \rangle^+$ play the role of the vacuum expectation value. The kinematical residue equation

$$-i \operatorname{Res}_{\theta=\theta'} F_{n+2}^{-+}(\theta + i\pi, \theta', \theta_1, \dots, \theta_n) = -2F_n^{-+}(\theta_1, \dots, \theta_n) \quad (\text{B.5})$$

connects either the even or the odd particle form factors to each other. The solution, starting with $G_0 = 1$ and $G_1(y) \equiv 1$ is given by

$$G_n = \frac{1}{y_{nn-1}} G_{n-2} + \text{perm} = \sum_{\text{all pairings}} \frac{1}{\prod_{\text{all pairs}(i,j)} y_{ij}} \quad (\text{B.6})$$

where $H_{2n} = (-2)^n$ and $y_{ij} = y_i + y_j$. Here we chose a slightly different normalization for both r^{-+} and H_{2n} from the ones in Section 3.1, but the form factors are the same.

B.2 Direct infinite volume calculation

The expansion of the free boson field with the Neumann or Dirichlet boundary conditions are

$$\Phi(x, t) = \begin{cases} \int_0^\infty \tilde{d}k \{ a_+(k) e^{-i\omega(k)t} + a_+^\dagger(k) e^{i\omega(k)t} \} \phi_k^+(x) & t < 0 \\ \int_0^\infty \tilde{d}k \{ a_-(k) e^{-i\omega(k)t} - a_-^\dagger(k) e^{i\omega(k)t} \} \phi_k^-(x) & t > 0 \end{cases}, \quad \phi_k^\pm(x) = e^{ikx} \pm e^{-ikx} \quad (\text{B.7})$$

where the creation/annihilation operators are normalized as

$$[a_\pm(k), a_\pm^\dagger(k')] = 4\pi\omega(k)\delta(k - k') \quad (\text{B.8})$$

The modes are orthogonal with a given boundary condition (3.24) and they form a complete system (3.25), so each basis can be expressed in terms of the other

$$\int_{-\infty}^0 \phi_k^{\pm*}(x) \phi_{k'}^\mp(x) dx = 2i \frac{(k+k') \mp (k-k')}{k^2 - k'^2} = 2i \frac{(k+k') \mp (k-k')}{\omega^2(k) - \omega^2(k')} \equiv A_{kk'}^{\pm\mp} \quad (\text{B.9})$$

As the quantum field, Φ , and its conjugate momentum, $\partial_t \Phi = \Pi$, is continuous in the bulk, we can relate the creation and annihilation operators of different boundary conditions to each other. Projecting $\Phi(x, t=0)$ and $\Pi(x, t=0)$ onto the modes and combining them results

$$\begin{aligned} a_+(k) &= \int_0^\infty \frac{i}{\pi} \frac{k' dk'}{\omega(k')} \left\{ \frac{a_-(k')}{\omega(k) - \omega(k')} - \frac{a_-^\dagger(k')}{\omega(k) + \omega(k')} \right\} \\ a_-(k) &= k \int_0^\infty \frac{i}{\pi} \frac{dk'}{\omega(k')} \left\{ \frac{a_+(k')}{\omega(k) - \omega(k')} + \frac{a_+^\dagger(k')}{\omega(k) + \omega(k')} \right\} \end{aligned} \quad (\text{B.10})$$

These are nothing but infinite dimensional Bogliubov transformations. The vacuum state of the Neumann boundary condition is a complicated coherent state for the Dirichlet boundary condition (3.31), and we parametrize it as

$$|0\rangle^+ = \mathcal{N} \left(1 + \int_0^\infty \tilde{d}k_0 K_1^{+-}(k_0) a_-^\dagger(k_0) \right) \exp \left\{ \frac{1}{2} \iint_0^\infty \tilde{d}k_1 \tilde{d}k_2 K_2^{+-}(k_1, k_2) a_-^\dagger(k_1) a_-^\dagger(k_2) \right\} |0\rangle^- \quad (\text{B.11})$$

where K_2^{+-} is symmetric in its arguments, and $\mathcal{N} = -\langle 0|0\rangle^+$. Now demanding $a_+|0\rangle^+ = 0$ constrains the form of the K_1^{+-} and K_2^{+-} kernels, which are the solutions of

$$0 = \int_0^\infty k' d\tilde{k}' \frac{1}{\omega(k) - \omega(k')} K_1^{+-}(k') \quad (\text{B.12})$$

and

$$-\frac{k'}{\omega(k) + \omega(k')} - \int_0^\infty d\tilde{k}_1 \frac{k_1}{\omega(k_1) - \omega(k)} K_2^{+-}(k_1, k') = 0 \quad (\text{B.13})$$

Or, the other way around, we can express the Dirichlet vacuum with Neumann

$$|0\rangle^- = \mathcal{N}^* \left(1 + \int_0^\infty d\tilde{k}_0 K_1^{-+}(k_0) a_+^+(k_0) \right) \exp \left\{ \frac{1}{2} \iint_0^\infty d\tilde{k}_1 d\tilde{k}_2 K_2^{-+}(k_1, k_2) a_+^+(k_1) a_+^+(k_2) \right\} |0\rangle^+ \quad (\text{B.14})$$

with K_2^{-+} being symmetric. The condition $a_-(k)|0\rangle^- = 0$ leads to

$$0 = \int_0^\infty k d\tilde{k}' \frac{1}{\omega(k) - \omega(k')} K_1^{-+}(k') \quad (\text{B.15})$$

and

$$\frac{1}{\omega(k) + \omega(k')} - \int_0^\infty d\tilde{k}_1 \frac{1}{\omega(k_1) - \omega(k)} K_2^{-+}(k_1, k') = 0 \quad (\text{B.16})$$

Solving the equations (B.12-B.16) from scratch is a demanding task, but we can still check that the prediction from the bootstrap approach does satisfy them.

B.2.1 Bootstrap predictions

Comparing the bosonic algebra (B.8) to the free boson Zamolodchikov-Faddeev algebra (A) shows that they only differ in the normalization, $Z(\theta) = \frac{1}{\sqrt{2}}a(k)$, with $k = m \sinh \theta$. Then we can relate the one-particle form factor to the K_1 kernel, as

$$F_1^{-+}(\theta) = \frac{1}{\sqrt{2}} -\langle 0|a_+^+(k)|0\rangle^+ = \frac{1}{\sqrt{2}} \mathcal{N} K_1^{-+*}(k). \quad (\text{B.17})$$

From the bootstrap approach we get

$$F_1^{-+}(\theta) = \mathcal{N} 2i \cosh \frac{\theta}{2} \quad (\text{B.18})$$

where \mathcal{N} plays the role of the ground state expectation value. To see that the resulting K_1^{-+} kernel satisfy (B.15) let us rewrite it in term of rapidity variables⁶,

$$\int_0^\infty \frac{d\theta'}{2\pi i} I_1(\theta'|\theta) = 0 \quad , \quad I_1(\theta'|\theta) = \frac{1}{\cosh \theta' - \cosh \theta} \cosh \frac{\theta'}{2}. \quad (\text{B.19})$$

By observing that

$$I_1(\theta'|\theta) = I_1(-\theta'|\theta) = -I_1(\theta' + 2i\pi|\theta) = -I_1(-\theta' + 2i\pi|\theta) \quad (\text{B.20})$$

we can extend the integration contour, depicted on Figure 3, and get

$$\int_0^\infty \frac{d\theta'}{2\pi i} I_1(\theta'|\theta) = \frac{1}{4} \oint_C \frac{d\theta'}{2\pi i} I_1(\theta'|\theta) = 0 \quad (\text{B.21})$$

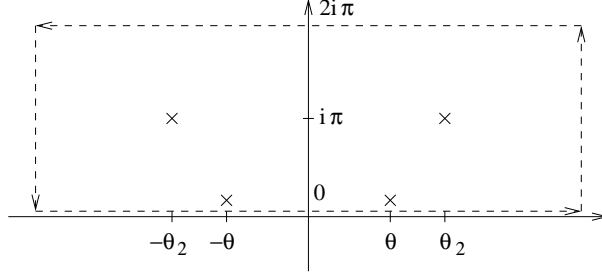


Figure 3: Contour of integration and poles of the integrand for checking the one particle term.

where in the last step we applied the Residue theorem and the cancellation of the residues.

Similarly, one can relate the K_1^{+-} kernel to the one particle form factor, as

$$F_1^{++}(\theta + i\pi) = \frac{1}{\sqrt{2}} \langle -|a_-(k)|+ \rangle = \frac{1}{\sqrt{2}} \mathcal{N} K_1^{+-}(k) \quad (\text{B.22})$$

Then, in the rapidity variables (B.12) takes the form

$$\int_0^\infty \frac{d\theta'}{2\pi i} J_1(\theta'|\theta) = 0 \quad , \quad J_1(\theta'|\theta) = \frac{\sinh \theta' \sinh \frac{\theta'}{2}}{\cosh \theta' - \cosh \theta}. \quad (\text{B.23})$$

Again, J_1 obeys the property

$$J_1(\theta'|\theta) = J_1(-\theta'|\theta) = -J_1(\theta' + 2i\pi|\theta) = -J_1(-\theta' + 2i\pi|\theta) \quad (\text{B.24})$$

Closing the contour as before and applying the Residue theorem proves (B.12).

In an analogous way one finds

$$F_2^{++}(\theta_1, \theta_2) = \frac{1}{2} \mathcal{N} K_2^{-+*}(k_1, k_2) \quad , \quad F_2^{++}(\theta_1 + i\pi, \theta_2 + i\pi) = \frac{1}{2} \mathcal{N} K_2^{+-}(k_1, k_2) \quad (\text{B.25})$$

with the bootstrap solution of the form factor axioms given as

$$F_2^{++}(\theta_1, \theta_2) = -4\mathcal{N} \frac{\cosh \frac{\theta_1}{2} \cosh \frac{\theta_2}{2}}{\cosh \theta_1 + \cosh \theta_2} \quad (\text{B.26})$$

The equations (B.16) and (B.13) takes the form

$$\begin{aligned} \frac{1}{\cosh \theta + \cosh \theta'} &= -4i \int_0^\infty \frac{d\theta_1}{2\pi i} I_2(\theta_1|\theta, \theta') \\ \frac{\sinh \theta'}{\cosh \theta + \cosh \theta'} &= 4i \int_0^\infty \frac{d\theta_1}{2\pi i} J_2(\theta_1|\theta, \theta') \end{aligned} \quad (\text{B.27})$$

with

$$\begin{aligned} I_2(\theta_1|\theta, \theta') &= \frac{1}{\cosh \theta_1 - \cosh \theta} \frac{\cosh \frac{\theta_1}{2} \cosh \frac{\theta'}{2}}{\cosh \theta_1 + \cosh \theta'} \\ J_2(\theta_1|\theta, \theta') &= \frac{\sinh \theta_1}{\cosh \theta_1 - \cosh \theta} \frac{\sinh \frac{\theta_1}{2} \sinh \frac{\theta'}{2}}{\cosh \theta_1 + \cosh \theta'} \end{aligned} \quad (\text{B.28})$$

⁶To avoid the pole singularity on the real line we used the previous ϵ - prescription.

satisfying

$$\begin{aligned} I_2(\theta_1|\theta, \theta') &= I_2(-\theta_1|\theta, \theta') = -I_2(\theta_1 + 2i\pi|\theta, \theta') = -I_2(-\theta_1 + 2\pi i|\theta, \theta') \\ J_2(\theta_1|\theta, \theta') &= J_2(-\theta_1|\theta, \theta') = -J_2(\theta_1 + 2i\pi|\theta, \theta') = -J_2(-\theta_1 + 2\pi i|\theta, \theta') \end{aligned} \quad (\text{B.29})$$

so that we can again close the contour as depicted on Figure 3. Applying the Residue theorem then proves (B.27).

To summarize, the predictions of the bootstrap approach,

$$\begin{aligned} K_1^{-+}(k) &= -i2\sqrt{2} \cosh \frac{\theta}{2} \quad , \quad K_2^{-+}(k_1, k_2) = -8 \frac{\cosh \frac{\theta_1}{2} \cosh \frac{\theta_2}{2}}{\cosh \theta_1 + \cosh \theta_2} \\ K_1^{+-}(k) &= -2\sqrt{2} \sinh \frac{\theta}{2} \quad , \quad K_2^{+-}(k_1, k_2) = -8 \frac{\sinh \frac{\theta_1}{2} \sinh \frac{\theta_2}{2}}{\cosh \theta_1 + \cosh \theta_2} \end{aligned} \quad (\text{B.30})$$

does satisfy the constraints derived directly in the field theoretical approach and thus provides an explicit relation between the incoming and outgoing vacua, up to an overall normalization.

B.3 Direct finite volume calculation

In this subsection we map our problem to the open/closed string vertex problem. In doing so we put the system in finite volume by introducing another boundary at $x = -L$ with Neumann boundary condition. Eventually we will take the limit $L \rightarrow \infty$, thus the boundary condition at $x = -L$ is irrelevant.

If the right boundary at $x = 0$ is chosen to be Neumann then the complete system, satisfying the equations of motion and the boundary conditions, is given as

$$f_{2n}^+(x) = \begin{cases} \sqrt{\frac{2}{L}} \cos(k_{2n}x) & n \in \mathbb{Z}^+ \\ \frac{1}{\sqrt{L}} & n = 0 \end{cases} \quad ; \quad k_{2n} = 2n \frac{\pi}{2L}. \quad (\text{B.31})$$

Have we chosen the right boundary to be Dirichlet, we would get the complete system

$$f_{2m+1}^-(x) = \sqrt{\frac{2}{L}} \sin(k_{2m+1}x) \quad , \quad m \in \mathbb{Z}_0^+ \quad ; \quad k_{2m+1} = (2m+1) \frac{\pi}{2L}. \quad (\text{B.32})$$

Thus for $t < 0$ we have even, while for $t > 0$ we have odd modes and they never coincide. They are normalized as

$$\langle f_{2n}^+ | f_{2n'}^+ \rangle = \delta_{nn'} \quad ; \quad \langle f_{2m+1}^- | f_{2m'+1}^- \rangle = \delta_{mm'} \quad (\text{B.33})$$

and they form separately a complete set

$$\sum_{n=0}^{\infty} f_{2n}(x) f_{2n}(y) = \delta(x-y) \quad , \quad \sum_{m=0}^{\infty} f_{2m+1}(x) f_{2m+1}(y) = \delta(x-y) \quad , \quad x, y \in [-L, 0] \quad (\text{B.34})$$

where we introduced the scalar product $\langle f|g \rangle = \int_{-L}^0 f(x)g(x)dx$. Their overlaps are

$$\langle f_{2m+1}^- | f_{2n}^+ \rangle \equiv A^{-+}(2m+1, 2n) = \langle f_{2n}^+ | f_{2m+1}^- \rangle \equiv A^{+-}(2n, 2m+1) = \begin{cases} \frac{\sqrt{2}}{L} \frac{k_{2m+1}}{\omega_0^2 - \omega_{2m+1}^2} & n = 0 \\ \frac{2}{L} \frac{k_{2m+1}}{\omega_{2n}^2 - \omega_{2m+1}^2} & n \geq 1 \end{cases}$$

The field obeys the mode expansion

$$\Phi(x, t) = \begin{cases} \sum_{n=0}^{\infty} \frac{f_{2n}^+(x)}{\sqrt{2\omega_{2n}}} (a_+(2n)e^{-i\omega_{2n}t} + a_+^+(2n)e^{i\omega_{2n}t}) & t < 0 \\ \sum_{m=0}^{\infty} \frac{f_{2m+1}^-(x)}{\sqrt{2\omega_{2m+1}}} (a_-(2m+1)e^{-i\omega_{2m+1}t} + a_-^+(2m+1)e^{i\omega_{2m+1}t}) & t > 0 \end{cases} \quad (\text{B.35})$$

with $\omega_n = \sqrt{m^2 + k_n^2}$. The commutation relations turns out to be

$$[a_+(2n), a_+^+(2m)] = \omega_{2n} \delta_{nm} \quad , \quad [a_-(2n+1), a_-^+(2m+1)] = \omega_{2n+1} \delta_{nm} \quad (\text{B.36})$$

Similarly to the infinite volume case we can relate the modes to each other by demanding the continuity the field and its momentum $\Pi = \partial_t \Phi$. Projecting $\Phi(x, t=0)$ and $\Pi(x, t=0)$ onto $\langle f_n^\pm |$ and combining them results in

$$\begin{aligned} a_+(2n) &= \sum_{m=0}^{\infty} \frac{A^{+-}(2n, 2m+1)}{2\omega_{2m+1}} \{ (\omega_{2n} + \omega_{2m+1}) a_-(2m+1) + (\omega_{2n} - \omega_{2m+1}) a_-^+(2m+1) \} \\ a_-(2m+1) &= \sum_{n=0}^{\infty} \frac{A^{-+}(2m+1, 2n)}{2\omega_{2n}} \{ (\omega_{2m+1} + \omega_{2n}) a_+(2n) + (\omega_{2m+1} - \omega_{2n}) a_+^+(2n) \} \end{aligned} \quad (\text{B.37})$$

and the conjugate relations. The compatibility of these relations is granted due to the unitarity

$$\begin{aligned} \sum_{m=0}^{\infty} A^{+-}(2n, 2m+1) A^{-+}(2m+1, 2n_1) &= \delta_{n, n_1} \\ \sum_{n=0}^{\infty} A^{-+}(2m+1, 2n) A^{+-}(2n, 2m_1+1) &= \delta_{m, m_1} \end{aligned} \quad (\text{B.38})$$

The states are built over a Fock vacuum defined as

$$a_+(2n)|0\rangle^+ = 0 \quad , \quad a_-(2n+1)|0\rangle^- = 0 \quad n = 0, 1, 2, \dots \quad (\text{B.39})$$

A multiparticle Neumann/Dirichlet state are generated by repeated action of creation operators on the corresponding vacuum state,

$$\begin{aligned} |\{n_1, \dots, n_N\}\rangle^+ &= a_+^+(2n_1) \dots a_+^+(2n_N) |0\rangle^+ \\ |\{m_1, \dots, m_M\}\rangle^- &= a_-^+(2m_1+1) \dots a_-^+(2m_M+1) |0\rangle^- \end{aligned} \quad (\text{B.40})$$

We are interested in the overlap of an incoming Neumann state and an outgoing Dirichlet state. To this end let us flip back the outgoing Dirichlet states to independent incoming ones by hermitian conjugation. To distinguish the flipped states from the original ones let us introduce a new set of bosonic operators as

$$a_+(2n) \mapsto c_1(2n) \quad , \quad a_-(2m+1) \mapsto c_2^+(2m+1) \quad (\text{B.41})$$

As the hermitian conjugation reverses the order of the operators, the new ones satisfy the algebra

$$[c_1(2n), c_1^+(2n')] = \omega_{2n} \delta_{nn'} \quad , \quad [c_2(2m+1), c_2^+(2m'+1)] = \omega_{2m+1} \delta_{m, m'} \quad (\text{B.42})$$

and the other commutators vanish. An incoming state is now built over the Fock vacuum defined as

$$c_1(2n)|0,0\rangle = 0 \quad , \quad c_2(2m+1)|0,0\rangle = 0 \quad (\text{B.43})$$

and the states are generated by repeated action of creation operators

$$|\{n_1, \dots, n_N\}, \{m_1, \dots, m_M\}\rangle = c_1^+(2n_1) \dots c_1^+(2n_N) c_2^+(2m_1+1) \dots c_2^+(2m_M+1)|0,0\rangle \quad (\text{B.44})$$

We define the vertex state $|V\rangle$ as

$$-\langle \{m_1, \dots, m_M\} | \{n_1, \dots, n_N\} \rangle^+ \equiv \langle V | \{n_1, \dots, n_N\}, \{m_1, \dots, m_M\} \rangle \quad (\text{B.45})$$

We parametrize it as

$$|V\rangle = \mathcal{N}^* e^\Delta |0,0\rangle \quad (\text{B.46})$$

with

$$\begin{aligned} \Delta = & \sum_{n_1, n_2=0}^{\infty} \frac{V_{++}(2n_1, 2n_2)}{2} \frac{c_1^+(2n_1) c_1^+(2n_2)}{\omega_{2n_1} \omega_{2n_2}} + \sum_{n, m=0}^{\infty} V_{+-}(2n, 2m+1) \frac{c_1^+(2n) c_2^+(2m+1)}{\omega_{2n} \omega_{2m+1}} + \\ & + \sum_{m_1, m_2=0}^{\infty} \frac{V_{--}(2m_1+1, 2m_2+1)}{2} \frac{c_2^+(2m_1+1) c_2^+(2m_2+1)}{\omega_{2m_1+1} \omega_{2m_2+1}} \end{aligned} \quad (\text{B.47})$$

and $\mathcal{N} = -\langle 0|0\rangle^+$. The $V_{\pm\pm}$ functions are called the Neumann coefficients, V_{++} and V_{--} are symmetric in their arguments.

After flipping, the relations (B.37) become

$$\begin{aligned} c_1(2n) - \sum_{m=0}^{\infty} \frac{A^{+-}(2n, 2m+1)}{2\omega_{2m+1}} \{(\omega_{2n} - \omega_{2m+1})c_2(2m+1) + (\omega_{2n} + \omega_{2m+1})c_2^+(2m+1)\} &= 0 \\ c_2(2m+1) - \sum_{n=0}^{\infty} \frac{A^{-+}(2m+1, 2n)}{2\omega_{2n}} \{-(\omega_{2n} - \omega_{2m+1})c_1(2n) + (\omega_{2n} + \omega_{2m+1})c_1^+(2n)\} &= 0 \end{aligned} \quad (\text{B.48})$$

and their hermitian conjugates, where the equations are understood in the weak sense, i.e. when sandwiched between the vertex state and any multiparticle state. These relations constrain the Neumann coefficients, resulting an overdetermined, nevertheless consistent system of equations. The only three independent ones are

$$\begin{aligned} \delta_{n, n_1} - \frac{1}{\omega_{2n_1}} \sum_{m=0}^{\infty} \frac{\omega_{2n} + \omega_{2m+1}}{2\omega_{2m+1}} A^{+-}(2n, 2m+1) V_{+-}^*(2n_1, 2m+1) &= 0 \\ \sum_{m_1=0}^{\infty} \frac{\omega_{2n} + \omega_{2m_1+1}}{\omega_{2m_1+1}} A^{+-}(2n, 2m_1+1) V_{--}^*(2m+1, 2m_1+1) + \\ &+ (\omega_{2n} - \omega_{2m+1}) A^{+-}(2n, 2m+1) = 0 \\ V_{++}^*(2n, 2n_1) - \sum_{m=0}^{\infty} \frac{\omega_{2n} - \omega_{2m+1}}{2\omega_{2m+1}} A^{+-}(2n, 2m+1) V_{+-}^*(2n_1, 2m+1) &= 0 \end{aligned} \quad (\text{B.49})$$

A similar problem was analyzed in the context of the open-closed string vertex [3, 4]. In the case when we choose Dirichlet boundary condition on the open string the resulting

equations (eqs. (2.34-2.36) of [3]) can be mapped to (B.48), thus we can simply read off the Neumann coefficients. Their volume dependence is encoded into some complicated modified gamma functions which, however, take a relatively simple form in the large volume limit. The large volume asymptotic solution reads as

$$\begin{aligned}
V_{+-}(2n, 2m+1) &= \frac{1}{L} \frac{k_{2m+1}}{\omega_{2n} - \omega_{2m+1}} \frac{\omega_{2m+1} + \omega_1}{\omega_{2n} + \omega_1} \frac{(\omega_{2n} + M)^{3/2}}{(\omega_{2m+1} + M)^{3/2}} + O(e^{-ML}) \\
V_{--}(2m_1+1, 2m_2+1) &= \frac{1}{L} \frac{k_{2m_1+1} k_{2m_2+1}}{\omega_{2m_1+1} + \omega_{2m_2+1}} \frac{\omega_{2m_1+1} + \omega_1}{(\omega_{2m_1+1} + M)^{3/2}} \frac{\omega_{2m_2+1} + \omega_1}{(\omega_{2m_2+1} + M)^{3/2}} + O(e^{-ML}) \\
V_{++}(2n_1, 2n_2) &= -\frac{1}{L} \frac{1}{\omega_{2n_1} + \omega_{2n_2}} \frac{(\omega_{2n_1} + M)^{3/2}}{\omega_{2n_1} + \omega_1} \frac{(\omega_{2n_2} + M)^{3/2}}{\omega_{2n_2} + \omega_1} + O(e^{-ML}) \quad (B.50)
\end{aligned}$$

where M is the mass of the particles. The unnormalized two-particle finite volume form factors are related to the Neumann coefficients as

$$\begin{aligned}
\langle -|a_+^\dagger(2n_1)a_+^\dagger(2n_2)|+\rangle &= \mathcal{N}V_{++}^*(2n_1, 2n_2) \\
\langle -|a_-(2m+1)a_+^\dagger(2n)|+\rangle &= \mathcal{N}V_{+-}^*(2n, 2m+1) \\
\langle -|a_-(2m_1+1)a_-(2m_2+1)|+\rangle &= \mathcal{N}V_{--}^*(2m_1+1, 2m_2+1) \quad (B.51)
\end{aligned}$$

We would like to take a sensible infinite volume limit $L \rightarrow \infty$, while keeping the momenta fixed, $k_{2n} = k$ and $k_{2m+1} = k'$. The dispersion relation does not change, $\omega_{2n} = \omega(k)$ and $\omega_{2m+1} = \omega(k')$. Comparing the completeness relations (3.25, B.34), the mode decomposition of the field (B.7, B.35) and the algebra relations (B.8, B.36) in finite and in infinite volume, one can read off the correct scaling of the mode operators,

$$\begin{aligned}
\sqrt{4L}a_+(2n) &\rightarrow a_+(k) \quad , \quad \sqrt{4L}a_+^\dagger(2n) \rightarrow a_+^\dagger(k) \\
-i\sqrt{4L}a_-(2m+1) &\rightarrow a_-(k') \quad , \quad i\sqrt{4L}a_-^\dagger(2m+1) \rightarrow a_-^\dagger(k') \quad (B.52)
\end{aligned}$$

In this infinite volume limit $\omega_1 \rightarrow M$, thus one gets

$$\begin{aligned}
-\langle 0|a_+^\dagger(k_1)a_+^\dagger(k_2)|0\rangle^+ &= -\mathcal{N}4 \frac{\sqrt{\omega(k_1)+M}\sqrt{\omega(k_2)+M}}{\omega(k_1)+\omega(k_2)} = -\mathcal{N}8 \frac{\cosh \frac{\theta_1}{2} \cosh \frac{\theta_2}{2}}{\cosh \theta_1 + \cosh \theta_2} \\
-\langle 0|a_-(k')a_+^\dagger(k)|0\rangle^+ &= -\mathcal{N}4i \frac{k'}{\omega(k)-\omega(k')} \frac{\sqrt{\omega(k)+M}}{\sqrt{\omega(k')+M}} = -\mathcal{N}8i \frac{\cosh \frac{\theta}{2} \sinh \frac{\theta'}{2}}{\cosh \theta - \cosh \theta'} \\
-\langle 0|a_-(k'_1)a_-(k'_2)|0\rangle^+ &= -\mathcal{N}4 \frac{k'_1 k'_2}{\omega(k'_1)+\omega(k'_2)} \frac{1}{\sqrt{\omega(k'_1)+M}\sqrt{\omega(k'_2)+M}} = \\
&= -\mathcal{N}8 \frac{\sinh \frac{\theta'_1}{2} \sinh \frac{\theta'_2}{2}}{\cosh \theta'_1 + \cosh \theta'_2} \quad (B.53)
\end{aligned}$$

Comparing the bosonic algebra (B.8) to the free boson Zamolodchikov-Faddeev algebra (A.2) shows that they only differ in the normalization, $Z(\theta) = \frac{1}{\sqrt{2}}a(k)$, thus the form factors are

$$\begin{aligned}
F_2^{-+}(\theta_1, \theta_2) &= -\langle 0|Z^+(\theta_1)Z^+(\theta_2)|0\rangle^+ = \frac{1}{2} - \langle 0|a_+^\dagger(k_1)a_+^\dagger(k_2)|0\rangle^+ \quad (B.54) \\
F_2^{-+}(i\pi + \theta', \theta) &= -\langle 0|Z^+(i\pi + \theta')Z^+(\theta)|0\rangle^+ = \frac{1}{2} - \langle 0|a_-(k')a_+^\dagger(k)|0\rangle^+ \\
F_2^{-+}(i\pi + \theta'_1, i\pi + \theta'_2) &= -\langle 0|Z^+(i\pi + \theta'_1)Z^+(i\pi + \theta'_2)|0\rangle^+ = \frac{1}{2} - \langle 0|a_-(k'_1)a_-(k'_2)|0\rangle^+
\end{aligned}$$

As \mathcal{N} plays the role of the vacuum expectation value, this result coincides with the solutions of the bootstrap axioms.

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